A PRACTICAL COURSE OF MATHEMATICS

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A PRACTICAL COURSE OF MATHEMATICS

GEOMETRY, ALGEBRA, PLANE AND SPHERICAL TRIGONOMETRY, CALCULUS, CONICS

BY

A. H. BELL, B.Sc.

Harling Scholar of Owens College, Manchester; formerly Principal of Sheerness Technical Institute

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PREFACE

This book includes the essentials of geometry and algebra, the fundamentals of plane and spherical trigonometry, elementary calculus, and a short introduction to conics.

The course is thus sufficiently comprehensive to meet the needs of most people, and will be particularly useful to engineers and other technicians, navigators, members of H.M. Forces and those preparing to enter such services.

Conforming with suggestions of the Ministry of Education in that the subject is treated as a whole and not in isolated parts, the book provides a course of mathematics on modern lines for technical and other post-primary schools.

It is hoped that the references to the pioneers may excite an interest in the history of mathematics—a history rich in great personalities and wonderful achievements, and inspired by the intense desire and striving of man to solve the mysteries of the universe and to appreciate the fullness of its glory.

While there may be no *royal* roads to mathematics, some seem to be more direct than others, and an endeavour has been made to conduct the reader along such roads. Academic treatment has been avoided as far as possible, practical demonstration being considered more in keeping with the spirit of the course and present-day requirements.

Many of the exercises are designed to prepare the way to subsequent chapters, and others to illustrate the applications of the relations established.

The chapters on calculus and conics are included not only for their usefulness but also to lead the student towards the realm of higher mathematics and to encourage him to go forward.

A. H. B.

CONTENTS

SYMBOLS AND ABBREVIATIONS

$=$	equals.	$\approx$	nearly equals.
$>$	greater than.	$<$	less than.
$\parallel$	parallel to.	$\perp$	perpendicular to.
$\rightarrow$	leading to.		
$+$	plus, or positive.	$-$	minus, or negative.
$\times$	multiplied by.	$\div$ or $/$	divided by
$\angle$ or $\wedge$	angle.	$\triangle$	triangle.
$^\circ$	degrees.	$\sqrt{}$	square root.
$!$	factorial product, e.g. $3! = 3 \times 2 \times 1$.		
$\propto$	varies as.	∞	infinity.
opp.	opposite.	int.	interior.
ext.	exterior.	rt. $\angle$	right angle.
$\log_{10}$	logarithm, base 10 (common)		
$\log_e$	logarithm, base e (Napierian)		
$\dfrac{dy}{dx}$	differential coefficient of y with respect to x.		
$\int$	integral.		

CHAPTER I

COMPUTATION, DECIMALS, METRIC SYSTEM, PERCENTAGE

1. The Decimal System.

It is assumed that the reader is familiar with the elementary principles and operations of the decimal system. Here, only a few special aspects and applications will be considered.

2. The Metric System.*

The metric system is a decimal system. Its operation is so simple that many people advocate its adoption by Britain, but although the system was for the second time legalized in Great Britain in 1897, in the opinion of others the cost and inconvenience that the change would entail outweighed other considerations.

3. Length.—The unit is the metre, originally taken to be a ten-millionth part of the quadrant of the meridian through Paris. It is approximately 39·37 in. The multiples and fractions of the unit follow in tenths reading from left to right, the fractions being placed after the decimal point. The names and sequence are shown below.

Kilo	Hecto	Deka	METRE		deci	centi	milli
1	3	2	7	•	4	6	5

The table is:

Multiples		*Fractions*	
10 metres	= 1 deka-m.	10 milli-m.	= 1 centi-m.
10 deka-m.	= 1 hecto-m.	10 centi-m.	= 1 deci-m.
10 hecto-m.	= 1 kilo-m.	10 deci-m.	= 1 metre.

The prefixes kilo (1000), hecto (100), deka (10) (Greek); deci ($\frac{1}{10}$), centi ($\frac{1}{100}$), milli ($\frac{1}{1000}$) (Latin), are used throughout the system. Macro (a million) and micro (a millionth) are met

* Enacted in 1799, and made compulsory by the French Republic in 1801.

occasionally. In contracted form, deka-m., hecto-m., kilo-m. are written Dm., Hm., Km.

4. Area.—The unit of area is the square whose side is a metre, and the first subdivision is a square whose side is 1 dm., i.e. a tenth of a metre. The square decimetre is therefore $\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$ of a square metre. In consequence, two places in the decimal system are required for square decimetres, and so on.

What the figures represent in the area 426·07381 sq. m. is shown below:

sq. Dm.	sq. m.		sq. dm.	sq. cm.	sq. mm.	
4	26	•	07	38	10	sq. m.

Note specially that the last figure represents 10 sq. mm. not 1 sq. mm. If 1 sq. mm. had been intended the last figure would have been replaced by 01.

A larger unit for land measurement is the ARE, which is equal to 100 sq. m. Its multiples and fractions are in tens and tenths respectively.

5. Volume.—The unit of volume is the cube whose sides are each a metre long. The first subdivision is the cubic decimetre, whose volume is $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{1000}$ of a cubic metre. Three places in the decimal system are required for cubic decimetres, and so on, as the number below illustrates:

c. m.		c. dm.	c.c.	c. mm.	
214	•	367	028	105	c. m.

Another unit, more convenient for many purposes, is the LITRE, which has a volume equal to one cubic decimetre. Its divisions are in tenths and its multiples in tens like those of length and therefore require only one place in the decimal system. A litre is about 1·76 pt.

6. Weight.—The unit is the gram, which is the weight of 1 c.c. of pure water at 4° C., at which temperature water has its greatest density.

The multiples and fractions are in tens and tenths repectively, as in the case of length.

The kilogram is about 2·2 lb. avoirdupois.

The choice of this unit of weight allows of ready calculations in *relative density* (the number of times a substance is as heavy as an equal volume of water), *weight* and *volume.*

EXAMPLE.—Find the weight of a bar of iron 5 cm. square and 20 cm. long and of relative density 7·8.

$$\text{Wt.} = 5 \times 5 \times 20 \times 7{\cdot}8 \text{ gm.}$$
$$= 3900 \text{ gm. or } 3{\cdot}9 \text{ Kgm.}$$

7. Change of Unit.—In any given number a new unit can be adopted by simply moving the decimal point. Thus for small volumes, as in chemical analysis, the cubic centimetre (c.c.) is generally used as the unit. For example, 26·5 c.c. means 26 c.c. and 500 c. mm. If the cubic metre had been retained as the unit this would have been written as 0·0000265 c. m.

8. Approximations.—Results are often required correct to a particular place in the system. For example, if 357·6 has to be stated to the nearest whole number, it is nearer 358 than 357. On the other hand, 357·4 is nearer 357 than 358. The only doubt is concerning ·5, and it is the practice to count it as greater than half for this purpose. Thus 357·5 to the nearest integer is 358.

The rule is, therefore, if the figure under consideration is less than 5 it is rejected; if 5 or more, the next higher place figure is increased by one.

EXERCISE I (A)

1. State to the nearest integer 105·3, 105·03, 105·8, 105·008, 105·67, 105·27.
2. State to the nearest second place of decimals, 39·367, 0·287, 0·497, 0·4907.

*Significant Figures.**

Opinions differ as to the precise definition of "significant figures". They have been defined as† "*those which must be retained for any position of the decimal point*", that is, whatever unit of the decimal system may be adopted. For example:

(i) ·0305 Km. = 30·5 m.

* The *Encyclopedia Britannica* states: "The significant figures are those which commence with the first figure other than zero in the number. Thus the significant figures of 13·027 and of ·00013027 are the same." This is hardly satisfactory as a definition.

† The Mathematical Association's Report on the Teaching of Arithmetic in Schools.

The first nought has become unnecessary, and is not a significant figure whereas the nought between the 3 and the 5 is retained, and like them is a significant figure.

(ii) 3050 m. = 3·05 Km.

The last nought has in the change become unnecessary, but again that between 3 and 5 is still necessary, and is a significant figure.

A nought at the end of a number may, however, be significant. For example:

3·050 m. may be taken to mean that the length is stated to the nearest millimetre, and is less than 3·0505 m. In this case, the nought is significant. To show that the length is expressed to the nearest final figure, a dot (or a number of dots) is sometimes placed on the line, thus 3·050. m.

Generally, all the figures of a number are significant except noughts which are immediately after the decimal point in a number less than one, and in many cases, but not all, the noughts at the end of a number.

9. Contracted Multiplication and Division.—In order to understand the following operations the student should convince himself of the accuracy of the following statements.

The first figure obtained by multiplying any denomination by:

(i) units, is of the same denomination, e.g. multiplying hundredths by units gives hundredths,

$$\cdot 01 \times 1 = \cdot 01;$$

(ii) tenths, is of the next lower denomination, e.g. multiplying hundredths by tenths gives thousandths,

$$\cdot 01 \times \cdot 1 = \cdot 001;$$

(iii) hundredths, is two lower in denomination, e.g. multiplying tenths by hundredths gives thousandths,

$$\cdot 1 \times \cdot 01 = \cdot 001.$$

On the other hand, multiplying by tens, hundreds, etc., raises the denomination.

Contracted Multiplication.—A common method is to adjust the numbers so that the multiplier is in standard form; that is, so that the first figure is a unit figure.

Example i.—Multiply 625·48 by 0·327 to the second place of decimals.

$625{\cdot}48 \times 0{\cdot}327$
$= 62{\cdot}548 \times 3{\cdot}27$

Make the first figure of ·327 a unit figure by multiplying by 10. To counterbalance this divide the multiplicand by 10.

```
   c ba
  62·548
     a bc
     3·27
 --------
 187·64
  12·51
   4·38
 ------
 204·53
```

Place the multiplier so that its unit figure (3) is under the place figure (4) to which the operation is to be worked.

Begin the first line with the product of these two figures (4 × 3) but the number to be carried forward from 8 × 3 must be added, making 14.

The first figure of the second line is obtained by the product of 2 and 5, the brought-forward number being 1, since 4 × 2 is nearer 10 than 0.

All lines begin with the same place, in this case the second. The figures lettered alike are the numbers to be multiplied to obtain the first figure of the line. Lettering avoids confusion.

Example ii.—Multiply 48·329 by 70·65 to first place of decimals.

$48{\cdot}329 \times 70{\cdot}65 = 483{\cdot}29 \times 7{\cdot}065$

```
 dcba
 483·29
     a bcd
     7·065
 ---------
 3383·0      First figure from (2 × 7) + 6 brought forward.
   29·0          ,,    ,,   ,,  (8 × 6) + 2     ,,      ,,
    2·4          ,,    ,,   ,,  (4 × 5) + 4     ,,      ,,
 ------
 3414·4
```

Contracted Division.—Divisor converted into standard form.

Example i.—Divide 36·7542 by ·7605.

$\dfrac{36{\cdot}7542}{{\cdot}7605} = \dfrac{367{\cdot}542}{7{\cdot}605}$

Multiplying both numbers by 10 does not alter the quotient.

```
           48·33
        --------
7·6̸0̸5̸)367·542
         304 20
         ------
          63 34  →
          60 84
          -----
           2 50
           2 28
           ----
             22
             23  →
```

The first figure (4) of the answer is placed above, in the place it would be put if got by short division by the unit figure (7) of the divisor.

Instead of bringing down the next figure (2) or adding a 0 in subsequent lines a figure is crossed off the divisor, in this case (5), and the method continued to the end.

(7 × 3) + 2 brought forward = 23, which is nearer 22 than (7 × 2) + 1.

The quotient is correct to the second decimal place. If more places are required the crossing off should be begun later so that at least one figure is left in the divisor for the last figure of the quotient.

EXAMPLE ii.—Divide 204·532 by 625·48 to the third place of decimals.

$$\frac{204{\cdot}532}{625{\cdot}48}=\frac{2{\cdot}04532}{6{\cdot}2548}$$ Dividing both numbers by 100.

$$\begin{array}{r} \cdot 327 \\ 6{\cdot}2\not{5}\not{4}\not{8})\overline{2{\cdot}045\not{3}\not{2}} \\ 1{\cdot}876 \\ \hline 169 \\ 125 \\ \hline 44 \\ 43 \\ \hline \end{array}$$

Two figures in each number may be crossed off at the outset.

EXERCISE I (B)

1. Convert the following:

(i) 1·356 m. into cm.
(ii) 387 cm. into m.
(iii) 25 mm. into m.
(iv) 2·2 Km. into m.
(v) 2·2 Km. into cm.
(vi) 3 sq. m. into sq. cm.
(vii) 5 sq. dm. into sq. m.
(viii) 3·25 hectares into ares.
(ix) 1538·5 c.c. into c. dm.
(x) 1·25 c.m. into c.c.
(xi) 2564 litres into hectolitres.
(xii) 4·8 Kgm. into gm.
(xiii) 67·5 mgm. into gm.

2. Taking the earth's circumference to be 25,000 miles, find the number of inches in a metre.

3. Calculate what fraction a kilometre is of a mile, taking a metre to equal 39·37 in.

4. Verify that, approximately, an inch equals 2·54 cm., and a metre, 1·1 yd.

5. A hectare is 100 ares. Compare the size of the hectare with that of the acre.

10. Mixed Fractions.

It is often labour-saving to use both decimal and vulgar fractions rather than to restrict the operations to one system of fractions.

For example, in computing $17{\cdot}28 \times 3{\cdot}125$ it is easier to use $3\frac{1}{8}$ instead of $3{\cdot}125$, thus:

$$\begin{array}{r} 17{\cdot}28 \\ \underline{3\frac{1}{8}} \\ 51{\cdot}84 \\ \underline{2{\cdot}16} \\ 54{\cdot}00 \end{array} \qquad \text{or} \qquad \overset{2{\cdot}16}{\cancel{17{\cdot}28}} \times \frac{25}{\cancel{8}} = \frac{216}{4} = 54.$$

The student should be on the alert for such opportunities, and should not plunge into calculations but make a preliminary examination of the numbers involved.

It is wise also to make an estimate of the answer—a rough approximation it may be in many cases, but generally it will be sufficiently near to ensure that the final answer is reasonable and does not contain a glaring error, say, in the position of the decimal point; e.g. $\frac{4}{\cancel{3}} \times \frac{\cancel{22}}{\cancel{7}} \times 3{\cdot}85 \times 3{\cdot}85$, by the rough cancelling shown, and treating $3{\cdot}85$ as equal to 4, can be seen to be about 64. Actually it is $62{\cdot}113$.

11. Percentages.

It is often convenient, especially for comparison, to reckon "per centum", that is "by hundred".

For example, one half is 50 out of a hundred, i.e. 50 per cent. The symbol for per cent is %. Thus $\frac{1}{2} = 50\%$, $\frac{1}{4} = 25\%$, and so on.

To convert a fraction into percentage form multiply it by 100.

EXAMPLES.—(i) $\frac{3}{4}$ is $\frac{3}{4} \times 100\% = 75\%$.
(ii) $\cdot 356$ is $\cdot 356 \times 100\% = 35{\cdot}6\%$.

To express a percentage in fractional form, divide by 100.

EXAMPLE.—35% in fractional form is $\frac{35}{100} = \cdot 35$ or $\frac{7}{20}$.

12. Error per cent.—Example: An error of ·05 of an inch is made in measuring (i) a line 2 in. long, and also (ii) in measuring a line 10 in. long. Express the errors per cent and compare them.

(i) Fractional error $= \dfrac{\cdot 05}{2}$.

$$\text{Error } \% = \frac{\cdot 05}{2} \times 100 = 2{\cdot}5\%.$$

(ii) Fractional error $= \frac{\cdot 05}{10}$.

$$\text{Error } \% = \frac{\cdot 05}{10} \times 100 = \cdot 5\%.$$

It is evident that, relatively, the first is the greater error.

Exercise I (c)

1. Find the errors per cent in taking (i) 3·14; (ii) $3\frac{1}{7}$, instead of 3·1416.
2. Find the error per cent in each of the following statements:
   (i) 1 m. $= 39\frac{1}{3}$ in.
   (ii) 1 m. $= 3\frac{1}{4}$ ft.
   (iii) 1 m. $= 1\cdot 1$ yd.
   (iv) 1 sq. m. $= 1\cdot 2$ sq. yd.
3. A cubic inch $= 2\cdot 54 \times 2\cdot 54 \times 2\cdot 54$ c.c. nearly. Work this out to the second decimal figure.
4. Compute $37\cdot 085 \times 0\cdot 528$ correct to the fourth significant figure.
5. Divide 643·3 by 12·566 to five significant figures.
6. Compute $\frac{3\cdot 14 \times 0\cdot 3536 \times 0\cdot 3536}{0\cdot 157}$.
7. Show that $\frac{83\cdot 69 \times 2\cdot 685 \times 0\cdot 384}{97\cdot 64 \times 0\cdot 067}$ is about 13.

   Find the value to 4 significant figures.
8. Compute $\cdot 6561 \times \cdot 8039 + \cdot 7547 \times \cdot 5948 \times \cdot 3256$.
9. Compute $\frac{\cdot 2994}{\cdot 9011 \times \cdot 6561}$.
10. The average of a set of numbers is their sum divided by their number. Find the average of the following five numbers: 3·142, 3·149, 3·136, 3·141, 3·14.
11. Find $3\frac{1}{16} \times 15\cdot 3624 \div 6\cdot 125$.
12. Find $\frac{93\cdot 07 \times 0\cdot 025}{3\frac{1}{7} \times 0\cdot 1428}$.

Note.—The sooner you learn to use logarithms, Chap. XVII, the more time you will save.

CHAPTER II

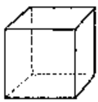
GEOMETRY

Material things occupy space, that is, take up room. They are said to have VOLUME. The solid things have shape or form, and in Geometry the features of the shapes of solids are studied.

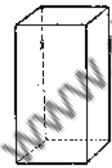
The word "Geometry" means "Earth measurement", and the earliest study of the subject was concerned with land measurement of the Nile district of Egypt.

1. Geometrical Forms.

The more important geometrical forms are illustrated in the figures shown (figs. 1 to 6). If actual models are available it is better to handle and examine them closely.

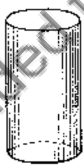


Cube
Fig. 1

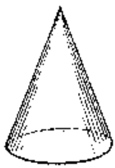


Square Prism
Fig. 2

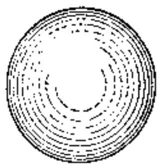
Square Pyramid
Fig. 3



Cylinder
Fig. 4



Cone
Fig. 5



Sphere
Fig. 6

Note carefully the following:

1. The boundaries of solids are *surfaces*. Some surfaces are flat or PLANE, others rounded or CURVED.

2. Edges or *lines* are the boundaries of surfaces. Some lines are straight, others curved.

3. The ends of lines are *points*.

2. Lines, Direction.

A line has length but not breadth. Its ends are points. If two lines intersect they do so at a point.

A *straight line* keeps the same direction all along its length. It has two directions, one being the exact opposite of the other.

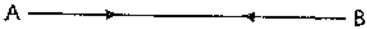


Fig. 7

If AB (fig. 7) is a straight line the direction from A to B is exactly opposite to that from B to A.

Another property often used is that the straight line distance is the shortest distance between two points.

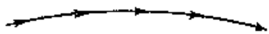


Fig. 8

A *curved line* changes its direction continuously (fig. 8).

3. Surfaces: Plane and Curved.

A **plane** or flat surface is straight in every direction. If a straight edge, say of a ruler, is placed on it, it will lie on the surface in every position.

If any two points be taken on a plane surface the straight line joining them will touch the surface at every point of its length.

If one plane surface is placed on another the two will touch at every point. There will be no space between them.

Test the surfaces of the models for planeness, using the straight edge of a ruler. See whether light passes through when the edge of the ruler is placed in various positions and directions.

It will be found in testing the curved surfaces of the **cylinder** and **cone** that although the straight edge can be placed along the surface in some positions it cannot be so placed in *all* positions.

In the case of the **sphere** there is no position in which the straight edge will lie along the surface.

Of the geometrical forms illustrated, the surfaces of the cube, square prism, and pyramid are all plane surfaces. The two ends of the cylinder and the base of the cone also are plane surfaces, but the remaining surface of the cylinder and of the cone and the whole surface of the sphere are curved.

All surfaces have both *length* and *breadth*, and their extent or area depends on these measurements or *dimensions* as they are often called.

4. Angles.

The difference in the directions OA and OB (fig. 9) is the angle AOB.

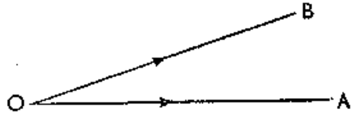


Fig. 9

Notice how the angle is named, the letter of the point or *vertex* being the middle of the three letters.

Straight Angle.

If a straight line OA is rotated until its direction from O to A is exactly reversed, it is rotated through a **straight angle.**

In fig. 10 A_1OA_2 is a straight angle if the direction OA_2 is exactly opposite to the direction OA_1. The rotation may be anti-

Fig. 10

clockwise or clockwise. (Clockwise means the way in which the pointers of a clock rotate.) Notice that A_1O and OA_2 are in a straight line.

Right Angle.

A right angle is half a straight angle. OA_1 has rotated through a right angle when it reaches the half-way position between OA_1 and OA_2.

A straight line at right angles to another is said to be **perpendicular** or **normal** to the other.

Cycle.—If the rotation of OA_1 is continued until it reaches its original direction, a *cycle* is completed.

A cycle equals *two* straight angles.

5. Classification of other Angles.

An *acute* angle is less than a right angle.

An *obtuse* angle is greater than a right angle but less than a straight angle.

A *reflex* angle is greater than a straight angle.

Acute

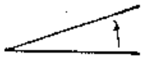


Fig. 11

Obtuse

Fig. 12

Reflex

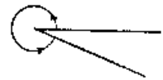


Fig. 13

The *complement* of an acute angle is the angle required to make up a right angle.

The *supplement* of an acute or an obtuse angle is the angle required to make up a straight angle.

6. The Protractor.

The protractor is an instrument used for measuring and marking out angles. It is usually made of metal, celluloid or wood. On it the straight angle is divided into 180 equal parts, each part being called a degree. Degrees are denoted by the sign °

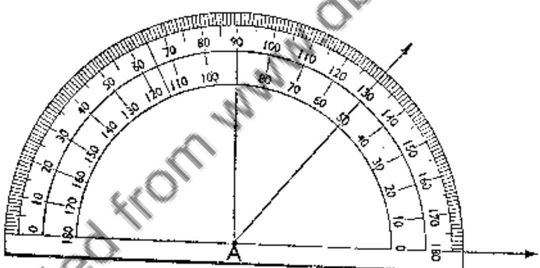


Fig. 14

Examine a protractor and notice that:

(i) The graduation marks are directed towards one point (A in fig. 14).

(ii) It is numbered from 0 to 180 both clockwise and anti-clockwise.

Fig. 14 shows how to use it.

Be sure that it is placed so that A is at the vertex of the angle and the appropriate 0 or zero mark over one of the lines of the angle.

7. Set-squares.

Set-squares are made of hard material, and have the shape of a triangle with one angle a right angle. The other angles are each 45° in one type, and 60° and 30° in another (figs. 15 and 16).

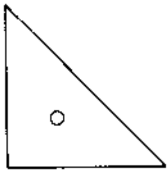


Fig. 15

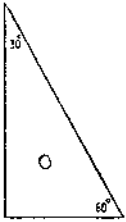


Fig. 16

These angles are set off on paper by drawing lines along the edges, the set square being held firmly in position.

Practise marking out these angles, and measure them with a protractor.

EXERCISE II (A)

1. The diagram (fig. 17) represents the chief compass directions for the centre point. What are the angles between the following directions measured (i) clockwise, (ii) anticlockwise: N. and E., N. and N.E., N. and S.E., N. and S.W., E. and N.W., S.E. and W., W. and N.E., N.W. and N.E.?

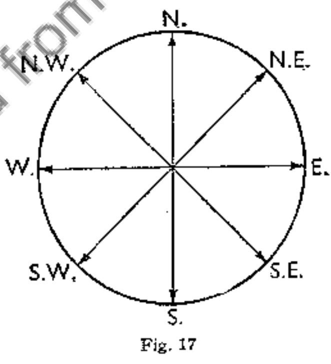


Fig. 17

2. Mark a point O and draw from it a straight line to represent the north direction. Draw straight lines from O to represent the directions (i) 30° E. of N.; (ii) 40° W. of N.; (iii) 20° E. of N.

3. How many degrees are there in (i) a right angle, (ii) a cycle?
4. Draw a straight line on transparent paper. Fold the paper so that one part of the straight line lies along the other. Make a sharp crease. Test the angles between the crease and the straight line for right angles.
5. A straight line AB is placed on another straight line CD so that end A is exactly on end C. What conclusions do you draw if end B falls (i) on end D, (ii) between C and D, (iii) beyond D?
6. Place a set-square on paper and without moving it draw a straight line along each edge. Turn the set-square through an angle, and again draw a straight line along each edge. Measure the angle between the two lines drawn along each of the three edges. How do the three angles compare?

Fig. 18

7. The straight line AB is rigidly joined to the straight line OA (fig. 18). If OA changes its direction by 20°, what is the change in the direction of AB? Draw the new position of OAB.
8. Draw any angle. To each of the containing lines draw a straight line at right angles. What do you know about the angle between these perpendiculars or normals? Draw a figure.

It is important to remember that the angle between the normals to two straight lines is equal to the angle between the straight lines.

FUNDAMENTAL APPLICATIONS

I. Intersecting Straight Lines. Opposite Vertical Angles.

Draw two straight lines AB and CD crossing at O (fig. 19). Mark the directions A to B and C to D.

It follows at once, since direction AO is the same as OB, and

CO as OD, that the difference between directions AO and CO is the same as between directions OB and OD:

i.e. $\angle AOC = \angle BOD.$

Similarly, $\angle AOD = \angle COB.$

These are called pairs of *opposite vertical angles*. Notice that they have the same vertex O.

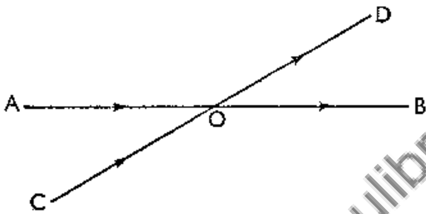


Fig. 19

EXERCISE.—Draw another straight line EF through O and make a list of the pairs of equal opposite vertical angles, naming the lines forming them in each case. Check by protractor measurement.

II. Parallel Straight Lines.

Parallel straight lines have the same direction but are not in a straight line with each other. In fig. 20, AB and CD both

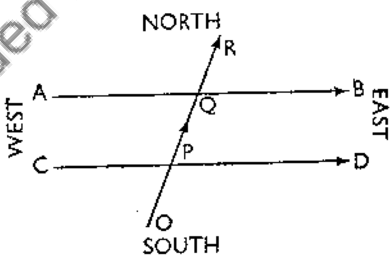


Fig. 20

point in, say, the easterly direction but are not in a straight line with each other.

If a straight line OPQR is drawn across them the difference between the direction of the parallels and of this transversal is the same for both;

i.e. $\angle BQR = \angle DPQ$, and $\angle AQR = \angle CPQ$.

There are other pairs of equal angles. Name them and test them in the following way. Take a piece of transparent paper and place it over one of the angles. Carefully make pencil marks over the lines limiting or bounding the angle. See if the traced angle exactly fits the other angle you consider equal to it.

Referring to the figure, the angles outside the parallels are called *exterior angles*, those between them, *interior angles*. Angles situated like $\angle AQP$ and $\angle QPD$ on different sides of the transversal, and having one bounding line in common, are said to be *alternate*.

You will find that *alternate interior angles* are equal;

i.e. $\angle AQP = \text{alt. } \angle QPD$

and $\angle CPQ = \text{alt. } \angle PQB$.

Exterior angles also are equal in pairs;

i.e. $\angle AQR = \angle OPD$

and $\angle OPC = \angle BQR$.

What do the interior angles BQP and QPD make together? Name other pairs of angles which together equal a straight angle.

Remember these facts concerning angles so formed and particularly the fundamental fact that an *exterior angle* like $\angle BQR$ is equal to its *opposite interior* angle, in this case $\angle DPQ$.

You will have noticed also that parallel lines neither converge nor diverge, that is they keep the same distance apart all along their length.

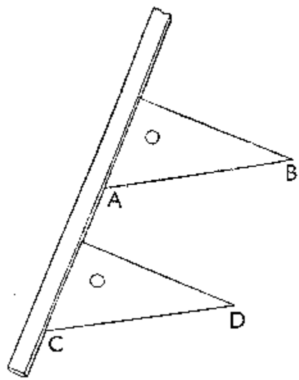


Fig. 21

Ruler and Set-square Construction of Parallel Straight Lines.

The edge of the ruler is used as a transversal and an angle of the set-square to give the direction of the parallels. Place the set-square in one of the positions indicated in fig. 21, and the ruler along one of its edges. Draw a line along one of the other edges, then slide the set-square along the ruler to another position. Draw another line along the same edge as before.

If it is required to draw a line parallel to a given straight line, place the set-square so that one edge is on the given line and proceed as before.

Accuracy greatly depends on keeping the ruler firmly in position. A little practice will give you ease in manipulation.

Exercise II (b)

1. Draw a straight line in any direction.

   Construct a pair of parallel straight lines making an angle of 60° with this line as a transversal.

   Measure all the angles in the figure. Measure also the distance the lines are apart at different points.

2. An exterior angle between one of a pair of parallel straight lines and a transversal is 30°. Make a diagram and on it show the size of the other angles.

3. If one angle formed by a transversal and parallel straight lines is a right angle, what are the other angles?

4. You will find squared paper convenient for this exercise and many others.

   Mark a point A, and for it show by arrows the N., E., S., and W. directions. On one side of A at a convenient distance mark a point B, and indicate for B the same four directions to agree with those for point A. Repeat the exercise for other points, and pick out the parallels.

III. The Circle.

The circle is such a useful figure that we will consider it at once.

It can be most readily drawn by means of a pair of compasses, consisting of two legs jointed so that they can be opened to various angles. One leg is pointed so that it can just pierce the paper, and the other leg is fitted with a pencil carrier.

The distance from the point of one leg to the pencil point on the other can be adjusted within limits.

With the compasses opened to, say, an inch between the points, describe a circle on a flat sheet of paper.

The centre point is called the **centre**; the boundary, the **circumference**; and a straight line from the centre to the circumference, a **radius** (plural radii).

All radii of the same circle are, of course, equal in length.

The names of parts of a circle, and of lines associated with a circle, are shown in fig. 22.

One of the great problems of mathematics is to find how the

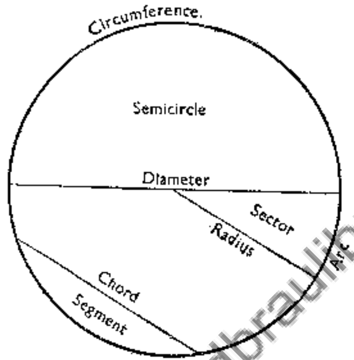


Fig. 22

length of the circumference of a circle can be calculated from the length of its diameter. This will be considered later.

Work the following simple exercises showing some uses of circles.

Exercise II (c)

1. Mark a point A on a sheet of paper. Now mark a large number of points an inch from A. Mark a few with crosses (×). It will be realized that in the circumference of a circle there is an infinite number of points.
2. Mark two points A and B $1\frac{1}{2}$ in. apart. Find and mark as many points as possible each of which is an inch from A and also an inch from B.
3. Continue Exercise 2 by finding other points equidistant from A and B, say 2 in., $1\frac{3}{4}$ in., $\frac{3}{4}$ in. Join the points and write down what you notice.
4. Use what you have learnt in Exercise 3 to divide a straight line $2\frac{1}{4}$ in. long into two equal parts, that is, to bisect it.
5. Join one of the points found in Exercise 3 to A and B, thus forming an angle. How does the straight line joining the equidistant points divide this angle? Test your conclusion.

6. Draw a circle of any convenient radius. See how many times this radius can be stepped round the circumference. Mark the points of the steps and join them to the centre of the circle. What is the angle between adjacent radii?

7. Draw any two intersecting circles and join the centres by a straight line. Note that the points of intersection are on opposite sides of this straight line. What happens when the distance between the centres is increased without altering the radius of each circle? Draw the case in which the circles just touch each other.

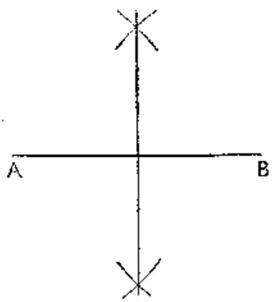


Fig. 23.—To bisect straight line AB

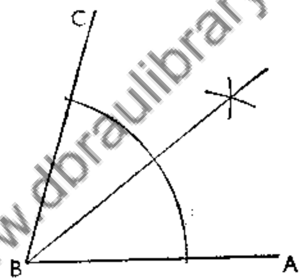


Fig. 24.—To bisect the angle ABC

8. Four important fundamental constructions are given (in figs. 23 to 26). Examine the diagrams carefully, making sure of the centres of the arcs, and reproduce them. Practise the constructions until you acquire accuracy.

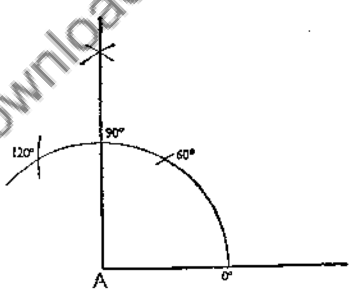


Fig. 25.—To draw from A a straight line at right angles to the straight line A0°

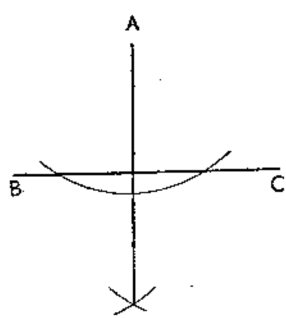


Fig. 26.—To draw from A a straight line at right angles to the straight line BC

IV. Plane Rectilineal Figures.

Plane rectilineal figures are plane surfaces bounded by straight lines. The following are important.

Triangle.—A triangle is bounded by three straight lines. As the name indicates, it has also three angles (fig. 27).

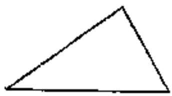


Fig. 27

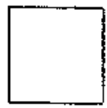


Fig. 28

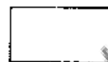


Fig. 29

Square.—A square has four sides, all equal in length, and all its angles are right angles (fig. 28).

If the side is an inch long, the area of the square is one square inch.

Oblong.—An oblong has four sides, those opposite being equal in length, and all its angles are right angles (fig. 29).

Construct one 3 in. long and 2 in. broad, and mark out squares showing that its area is $3 \times 2 = 6$ sq. in.

The square and oblong are often called **Rectangles.**

Fig. 30

Fig. 31

Parallelogram.—A parallelogram has four sides, the opposite sides being parallel (fig. 30).

Rhombus.—A rhombus has four sides, all equal, but its angles are not right angles (fig. 31).

Examine the figures named above and see if you can discover any other properties.

V. Angle between Planes.

Two plane surfaces intersect in a straight line and the angle between the surfaces is taken to be the angle between two straight lines, one in each plane, drawn at right angles to the line of intersection from the same point in it.

In the perspective figure (fig. 32), AB is the line of intersection of planes P and Q, CD in plane P is at right angles to AB, and

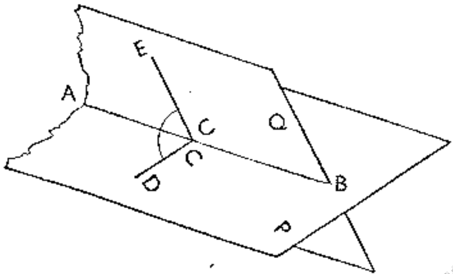


Fig. 32

CE in plane Q is at right angles to AB. The angle between the planes is ∠ECD. The angle is the same for all positions of C in AB.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS AND SIGNS

1. Possessions and Debts.

A man (A), after paying all his debts, may find that he still possesses a sum of money or something equivalent to a sum of money. Another man (B) may find that, although able to discharge all his debts, he has nothing left. A third man (C) may find that, after paying off as many debts as he is able, he has still debts to pay.

If we agree to call the state of A **positive,** as is usual, then the state of C is **negative,** and that of B neither positive nor negative. Suppose that A's possession is £15, then in Algebra his state is shown as +£15 (spoken, **plus** fifteen). The mark + is called the **plus sign,** and +15 is called a **positive** number.

If C's remaining debts amount to £15, his state is denoted by −£15 (spoken, **minus** fifteen). The mark − is called the **minus sign,** and −15 is called a **negative** number. It will be observed that the state of C is precisely **opposite** to that of A.

The state of B is represented by the **zero** figure, 0. Notice that in the zero state, there is an absence of both possessions and debts.

Note.—A number with no sign is considered positive.

Exercise III (a)

Answer the following:

1. After paying all debts, a man is worth £250. Represent his state algebraically, and contrast it with that of a man who owes £250.
2. After paying as many of his debts as possible, a man finds that he still owes £60. Represent his state algebraically.
3. Referring to Exercise 2, how much must the man earn before he is able to reach the zero state?
4. A man has £350, but his debts amount to £235. What is his actual state? Represent it algebraically.
5. Another man has £235, and his debts amount to £350. What is his actual state? Represent it algebraically.
6. A merchant, whose assets (possessions) amount to £350 and his debts to £200, joins in business with another merchant, whose assets amount to £500 and whose debts are £300. What are the assets and debts of the partnership? Represent its actual state algebraically.
7. A merchant, whose assets amount to £250 and whose debts are £350, joins in business with another merchant, whose assets are £350 and whose debts are £250. What is their joint state?
8. A merchant whose assets amount to £250 and debts to £350, joins in business with another, whose assets are £150 and debts £200. What is their joint state?

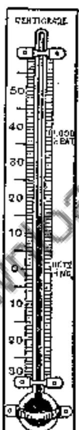


Fig. 1

2. Thermometer Scales.

Examine carefully the scale of a Centigrade thermometer. If possible, obtain one thermometer in which

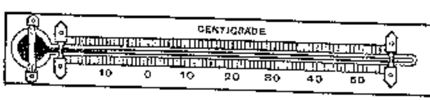


Fig. 2

the tube is vertical, as in fig. 1, and another in which it is horizontal, as in fig. 2.

Observe that the divisions appear to be equal, and that they are numbered from a mark numbered 0. This is the **zero** mark.

In fig. 1, the numbers above the zero are said to be positive, and those below the zero, negative.

In fig. 2, the numbers to the right of the zero are positive, and those to the left, negative.

You probably know that the temperature is given by the position of the surface of the liquid in the thermometer, as shown by the scale. Thus in fig. 1, since the surface is opposite the 15 mark above the zero, the temperature is $+15°$ (the sign ° denotes degrees). In fig. 2, the surface is opposite the 10 mark to the left of the zero, and the temperature indicated is $-10°$.

Exercise III (b)

Answer the following questions referring to thermometer scales:

1. Represent algebraically the temperature when the surface of the liquid is:
   (*a*) 12 divisions above the zero.
   (*b*) 5 divisions below the zero.
   (*c*) at the zero.
   (*d*) 100 divisions above the zero.
   (*e*) 15 divisions below the zero.
2. The reading of a thermometer is $+10°$. If the temperature rises 15°, what is the final reading?
3. The temperature falls from $+25°$ through 15°. What is the final reading?
4. The temperature falls 25° from $+15°$. What is the final reading?
5. The temperature rises through 20° from $-5°$. What is the final reading?
6. The temperature is $-10°$. Through how many degree divisions must the surface rise in order to reach zero?
7. The zero of a thermometer is lowered through 5 degree divisions. What will be the new numbering of the graduations originally numbered, $+15$, -10, -5, -2, 0?
8. Water freezes at 32° F. and boils at 212° F. How many degrees are there between these temperatures?
9. Alcohol freezes at $-112°$ C. and boils at 78° C. How many degrees are there between these temperatures?

3. Linear Measurements.

Draw a straight line BA (fig. 3), and in it mark a point O. If measurements made in the direction OA are positive, then

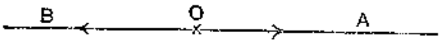


Fig. 3

measurements made in the opposite direction OB are negative.

Notice that the direction from B to A is the same as the direction OA, and the direction from A to B as the direction OB.

EXERCISE III (c)

1. From a point O, in a straight line, mark off the following distances in the directions indicated by the signs:

   $+3$ in., -4 in., $-6{\cdot}2$ in., $+4{\cdot}7$ cm., $-5{\cdot}8$ cm.

2. From a point O, in a straight line, mark off OA equal to $+5{\cdot}7$ cm., then from A mark off a distance AB, $-3{\cdot}4$ cm. (fig. 4). Represent algebraically the distance OB.

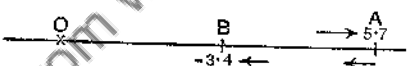


Fig. 4

3. Draw a straight line, and in it mark a zero point. Measure off a distance OB, $-4{\cdot}3$ cm., and from B a distance BC, $-3{\cdot}6$ cm. What is the distance OC? (Remember the sign.)
4. From a point O, in a straight line, mark off OA, $+3{\cdot}4$ cm., and from A, mark off AB, -6 cm. What is the distance OB? Why is OB negative?

4. Rotation.

Imagine the straight line OA (fig. 5) to be pivoted at O, and to be rotated without leaving the surface of the paper.

The line may be rotated either in the direction in which the hands of a clock move (clockwise), or in the contrary direction (counter-clockwise or anti-clockwise).

If anti-clockwise is taken to be positive rotation, as is usual, then clockwise is negative rotation.

Looked at from the North Pole, the direction of rotation of the Earth is positive (see fig. 6).

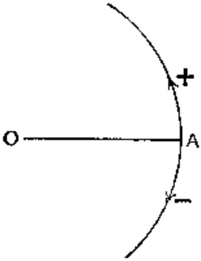


Fig. 5

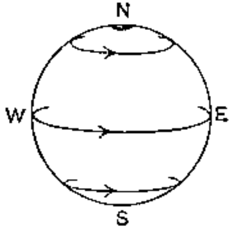


Fig. 6

Observe that the direction of the line OA changes as the line rotates, and is completely reversed when OA has rotated through

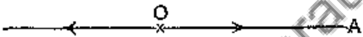


Fig. 7

half a revolution, i.e. 180° or a straight angle, in either the positive or negative direction (fig. 7).

Exercise III (d)

1. Mark a direction OA. Then, using a protractor, mark a direction OB +35° from the direction OA, and another, OC, −25° from OA. Measure the angle BOC.
2. From a standard direction OA, mark a direction OB, +80°. Now mark another direction, OC, −60° from OB.

   How many degrees is the direction OC from the direction OA?
3. Draw the compass directions from a point O, as shown in fig. 8.

   From OE mark off an angle +50°; from OW, an angle −35°; from OS, an angle −65°; and from ON, an angle 125°.

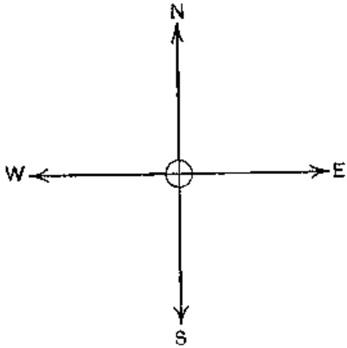


Fig. 8

4. Sea level being the zero, express algebraically the height of a mountain 3028 ft. high, and the depth of a mine 150 fathoms deep. (A fathom is 6 ft.)

5. Two equal toothed wheels are geared together. The first makes 60 revolutions per minute in the counter-clockwise direction. What is the direction of rotation of the second wheel? Represent the speeds algebraically.

6. Two pulleys are connected by a belt. The first makes 100 revolutions per minute in a clockwise direction, and the other twice as many revolutions per minute. Represent these speeds algebraically.

7. One clock (A) is 15 min. fast, and another (B) is 10 min. slow. Represent the errors algebraically.

8. One clock (A) gains 8 min. a day, another clock (B) loses 5 min. a day. Represent these changes algebraically.

9. A body falling to earth gains speed at the rate of 32·2 ft. per second each second. A body projected upward from the earth loses speed at the rate of 32·2 ft. per second each second. Represent these changes in speed algebraically.

10. When a spring balance is pulled, the backward force of the spring is felt. If the reading is 12 lb., represent algebraically the value of the pull, and that of the backward force of the spring.

11. A spiral spring can be stretched or compressed. How would you distinguish algebraically between these changes?

12. Represent 55 B.C. and 1915 A.D. algebraically.

CHAPTER IV

SUM AND DIFFERENCE

1. The pupil is warned that the conception of addition and subtraction usually formed from the rules of Arithmetic is not complete.

In Algebra, a much wider view of these processes is taken, and it will be seen that in finding the sum of numbers it is sometimes necessary to subtract the figures. It depends upon the sign of the numbers.

The teacher and pupils should construct in wood or cardboard the apparatus illustrated in fig. 1, which consists of two like scales with equal divisions numbered as shown.

One scale, A, is fixed and the other, B, movable. The two scales are contained in a frame F.

If the material mentioned is not available, the pupils should make a similar contrivance in their exercise books, the movable

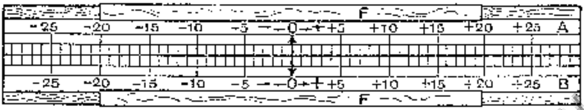


Fig. 1

scale being on a strip of paper passed through loops formed by making two parallel cuts through the page with a penknife.

On both scales, it will be seen that the positive numbers are on the right, and the negative numbers on the left of the zero mark.

With the help of these scales, it is proposed to establish some very important facts which should be remembered.

Example i.—*Find the sum of* $+9$ *and* $+5$.

Find the $+9$ mark on the fixed scale, then bring the zero of the movable scale opposite this $+9$ mark, and find the number on

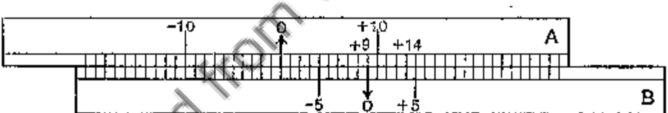


Fig. 2

the fixed scale to which the $+5$ on the movable scale is opposite. You will find it to be $+14$ (fig. 2).

Of course you know this to be correct.

If this exercise had been done by drawing a line 9 cm. long, and then extending it by 5 cm., it would have been noticed that the second line (5 cm.) followed from the end of and in the same direction, namely the positive direction, as the first line (9 cm.).

In the following exercises, the second number is added to the first by proceeding from the end of the length representing the first, in the direction indicated by the sign of the second number.

Example ii.—*Find the sum of* -9 *and* -5.

Find the -9 mark on the fixed scale; bring the zero of the movable scale to this mark, and find the number on the fixed scale to which the −5 on the movable scale is opposite.

You will find that the result is −14.

Example iii.—*Find the sum of* +9 *and* −5.

Find the +9 mark on the fixed scale, then move the lower scale until the 0 is opposite this mark. The number on the fixed scale to which the −5 of the lower scale is opposite gives at once the result, namely +4 (fig. 3).

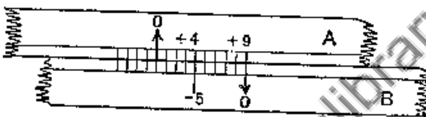


Fig. 3

In such a case you see that, to find the sum, the numbers, regardless of sign, are actually subtracted.

Observe that the result is positive, because the number with the plus sign is greater than that with the minus sign.

Notice that the −5 *is measured in the direction opposite to that in which the* +9 *is measured.*

Example iv.—*Find the sum of* +5 *and* −9.

Find the +5 mark on the fixed scale, move the lower scale until its zero mark is opposite the +5; then the number on the upper scale to which the −9 on the lower is opposite, is the sum of +5 and −9. The result is −4.

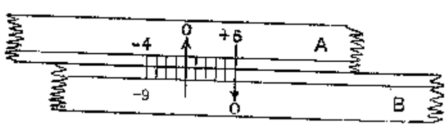


Fig. 4

Observe that the result is negative, because the number with the minus sign is greater than the number with the plus sign (fig. 4).

Note.—The sign + stands for addition also.

Thus 8+(−5) means, to +8 *add* −5.

The small bracket is used to separate the two signs.

EXERCISE IV (A)

Using the scales as in the foregoing examples, find the sum of:

1. $+15$ and $+6$. 2. -3 and -8. 3. -8 and $+3$.
4. $+12$ and -7. 5. -4 and $+11$. 6. -1 and $+1$.

Without using the scales, write down the answers to the following:

7. Find the sum of $+8$ and $+5$, and of 4 and -9.

Add, that is, find the sum of,

8. -8 and -5. 9. $+8$ and -5. 10. -8 and $+5$.

You will now see the reason for the following rules for finding the *sum* of two algebraic numbers.

(1) If the *signs* of the numbers are *alike*, *add* the numbers and prefix the sign of the numbers.

(2) If the *signs* of the numbers are *unlike*, *subtract* the smaller number from the greater, regardless of signs, then prefix the *sign* of the *greater* number.

EXERCISE IV (B)

Using the foregoing rules, find the sum of:

1. $+12$ and $+8$. 2. -12 and -8. 3. $+12$ and -8.
4. -12 and $+8$. 5. -20 and -5. 6. -20 and $+5$.
7. $+20$ and -5. 8. $+20$ and $+5$.
9. Using the scales, show that the result of adding $+8-4+3-5$ in order is the same as adding $+8+3-4$ and -5 in order, and as adding $+11$ and -9, i.e. the sum of the positive numbers and the sum of the negative numbers.

Find the value of the following:

10. $23-14+8-6-2+7$. 11. $-14+3-16-2+10$.
12. A ship sails E. for 2 hr. at 12 miles per hour, turns and sails W. for $3\frac{1}{4}$ hr. at 15 miles per hour, then turns again and sails E. for $1\frac{1}{4}$ hr. at 10 miles per hour. How far, and in what direction, is it from the starting-point?

2. Difference.

Being asked the difference between 5 and 9, almost everybody would say at once, four. In Algebra, however, this answer is incomplete, because the difference depends upon whether we view the difference from the 5 or from the 9.

Work the exercise on the special apparatus.

The movable scale is now used to measure differences.

EXAMPLE i.—Find the marks +5 and +9 on the fixed scale, move the lower scale until its 0 is opposite +5 on the upper scale; then the reading on the lower scale, which is opposite +9 on the upper, measures the difference between 5 and 9 when 5 is the number from which we reckon the difference. The answer is, of course, 4, meaning +4.

This exercise could have been stated as follows:

From 9 *subtract* 5. *Answer*, +4.

Or the exercise might have been stated as "*What does* 9 *become when* 5 *is made the zero?*"

Another way of stating the same exercise is, "*What must be added to* 5 *to make* 9?"

Measure the amount by placing the lower scale so that its zero is at 5.

It is seen that subtraction is the reverse process of addition.

Compare this result with that of Example iii, p. 28.

You see that the result is the same as when 9 and −5 are added.

EXAMPLE ii.—Place the lower scale so that the zero is opposite the 9 of the upper scale, and see what reading on the lower scale is opposite the 5 of the upper scale (fig. 5).

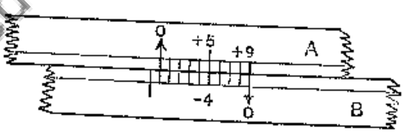


Fig. 5

The answer is −4. That is, −4 is the difference between 9 and 5 when regarded from the 9.

The exercise might have been stated as follows:

(i) *From* 5 *subtract* 9. *Result*, −4.

(ii) *What must be added to* 9 *to give* 5? *Answer*, −4.

Compare this result with that of Example iv, p. 28.

You see that the result is the same as when −9 and 5 are added.

Example iii.—*From* -9 *subtract* -5.

In other words, find the difference between -9 and -5 regarded from -5.

Find these positions on the upper scale.

Move the lower scale until the zero is at -5, and measure the difference. *Answer*, -4.

Comparing with Example iv, p. 28, it is seen that the result is the same as that obtained by adding $+5$ and -9.

Example iv.—*From* -5 *subtract* -9.

Placing the zero of the lower scale at -9 of the upper, the result is seen to be $+4$.

Example v.—*From* -9 *subtract* $+5$.

Proceed as before. See that the zero of the lower scale is at $+5$. The answer is -14, which is the same as that obtained by adding -5 to -9.

Example vi.—*From* $+5$ *subtract* -9.

Placing the zero of the lower scale at -9, the result is found to be $+14$ (fig. 6)

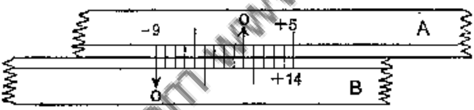


Fig 6

Summing up these exercises, it is seen that:

(i) *Subtracting a positive number is equivalent to adding a negative number.*

(ii) *Subtracting a negative number is equivalent to adding a positive number.*

The rule for subtraction is: *Change the sign of the number to be subtracted, and then proceed as in finding the sum.*

The following examples are interesting, because they illustrate this change in sign.

Example i.—*From* 0 *subtract* 5.

Place the zero of the lower scale at 5, and find to what reading on the lower scale the 0 of the upper is opposite. *Answer*, -5.

In other words, -5 must be added to $+5$ to give 0; or, if the zero is moved to 5, the original 0 becomes -5.

Example ii.—*From 0 subtract* -5.

Proceeding as before, the answer will be found to be $+5$.

You will notice that moving the zero to the -5 mark reverses the direction of the length from 0 to -5 (fig. 7).

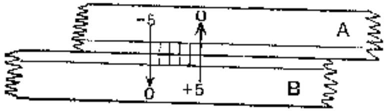


Fig. 7

Note.—The minus sign is the sign for subtraction also. Thus, $9-(-5)$ means, "From 9 subtract -5." We have seen that this is equal to $9+5$.

Exercise IV (c)

Verify each result by working the exercise on the special apparatus.

Subtract:

1. 6 from 10.
2. -6 from -10.
3. -10 from 6.
4. 10 from -6.
5. -10 from -6.
6. 10 from 6.
7. -6 from 10.
8. 6 from -10.

By how much does:

9. -18 differ from 10?
10. -18 from -10?
11. 18 from 10?
12. 18 from -10?

Subtract:

13. 5 from 0.
14. -5 from -5.
15. -3 from -7.
16. 5 from -5.
17. -5 from 5.
18. -7 from 3.

Find the value of the following:

19. $9-(-4)$.
20. $4-(-9)$.
21. $-3-(-8)-2$.
22. $0-(-5)$.
23. $0-(+5)$.
24. $9-(2-7)$.
25. $6-2+4-(5+3-1)$.
26. $6-(4+2-9)$.
27. $32-15-7+6-(-4)$.
28. $-(3-6+5)-(-2+8-4)$.

MULTIPLICATION AND DIVISION OF POSITIVE AND NEGATIVE NUMBERS

3. Multiplication.

Example i.— $+4$ *multiplied by* $+2$.

This may be taken to mean, on our scale, two consecutive lengths, each $+4$.

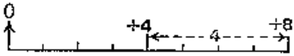


Fig. 8

That is, a length 8 (fig. 8).

It is also a short way of writing two fours added, i.e. $4+4$.

It will be readily understood that the sum of any number of positive numbers is positive. Thus, 5 fours added give $+20$.

The product of two positive numbers is positive.

Example ii.— -4 *multiplied by* $+2$.

This may be taken to mean two consecutive lengths, each -4. That is, a length -8 (fig. 9).

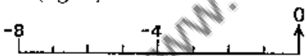


Fig. 9

It is also a short way of writing two minus fours added, i.e.

$$-4+(-4)=-4-4=-8.$$

Example iii.— $+4$ *multiplied by* -2.

This may be taken to mean four minus twos added, i.e.

$$-2+(-2)+(-2)+(-2)=-8.$$

Or that twice four has to be a subtracted number. Thus, from 0 subtract twice four.

$$0-2\times4 \text{ equals } 0-8, \text{ i.e. } -8.$$

The product of a positive and a negative number is negative.

The operations 4×2 and -4×2 are contrasted graphically in fig. 10, in which AC is 4, AB 4×2, and AD -4×2.

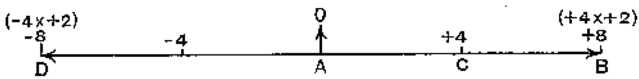


Fig. 10

Observe that 4×2 is changed into -4×2 by turning AB round A until its direction is reversed.

Example iv.— -4 *multiplied by* -2.

This may be taken to mean that twice minus four has to be a subtracted number. Now twice minus four is minus eight, and subtracting minus eight is equivalent to adding plus eight.

As an example of this, consider the following:

From 0 *subtract twice minus* 4.

$$0-(-8)=0+8=+8.$$

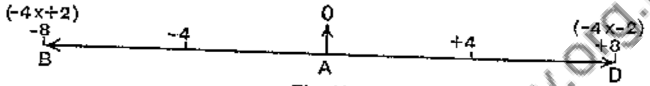


Fig. 11

If AB (fig. 11) represents -4×2, then -4×-2 is obtained by turning AB about A until its direction is reversed, i.e. until it is positive.

The product of two negative numbers is positive.

4. These examples, being fundamental, are so important that we shall illustrate them in another manner.

A product can be represented by the area of a rectangle or oblong. The area of such a figure is found by multiplying the length by the breadth.

Example i.— $+4\times +2$.

If we mark out the length $+4$ in a horizontal direction to the right, and from the end draw in the upward vertical direction

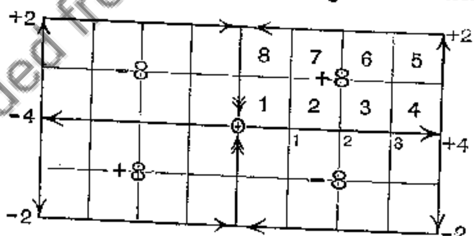


Fig. 12

the breadth $+2$, as shown in fig. 12, we turn in the anti-clockwise or positive direction of rotation.

The boundaries of the rectangle can be followed round in the anti-clockwise direction. We can agree to call an area, enclosed by boundaries described in this way, positive,* and so illustrate that the product of $+4$ and $+2$ is $+8$.

* This convention is used in Engineering.

EXAMPLE ii.— $-4 \times +2$.

The boundaries of the rectangle in this case are described in the *clockwise* or *negative* direction. The result is therefore -8 (fig. 12).

EXAMPLE iii.— $+4 \times -2$.

The figure shows the result to be -8.

EXAMPLE iv.— -4×-2.

The figure shows that since the area is described in the *anti-clockwise* or *positive* direction, the result is $+8$.

5. Summing up these results, we have:

The product of *two positive* numbers is *positive.*

The product of a *positive* and a *negative* number is *negative.*

The product of *two negative* numbers is *positive.*

This is often stated briefly in the form:

Two like signs give plus; two unlike signs, minus.

In signs:

+ multiplied by + = +, + multiplied by − = −,

− multiplied by − = +, − multiplied by + = −.

EXERCISE IV (D)

1. 6×3. 2. 6×-3. 3. -6×3. 4. -6×-3.
5. 1×1. 6. -1×1. 7. 1×-1. 8. -1×-1.
9. $\frac{3}{4} \times -\frac{1}{2}$. 10. $-1\frac{1}{2} \times -\frac{2}{3}$. 11. 0×3. 12. 0×0.
13. -2×0. 14. $-3{\cdot}2 \times 5$. 15. $3 \times -6 \times 5$.
16. $4 \times -5 \times -3$. 17. $-3 \times -6 \times -2$. 18. -2×-2.
19. $-2 \times -2 \times -2$. 20. $5 \times -3 \times -6 \times 2 \times -4\frac{1}{2}$.
21. A clock gains 5 min. a day. If it is correct now, how much will it be wrong 3 days hence, and how much was it wrong 3 days ago? Another clock loses 5 min. a day. If it is correct now, how much will it be wrong 3 days hence, and how much was it wrong 3 days ago?

    Select signs for gain and for loss, and for fast and for slow, and represent these four results algebraically.
22. What is the sign of the product of:

    (i) An odd number of negative numbers?

    (ii) An even number of negative numbers?

6. Division.

Division is the inverse of multiplication.

Examples.

(i) Since $+4$ multiplied by $+2$ gives $+8$,
therefore $+8$ divided by $+2$ gives $+4$,
and $+8$ divided by $+4$ gives $+2$.

(ii) Since -4 multiplied by $+2$ gives -8,
therefore -8 divided by $+2$ gives -4,
and -8 divided by -4 gives $+2$.

(iii) Since -4 multiplied by -2 gives $+8$,
therefore $+8$ divided by -2 gives -4,
and $+8$ divided by -4 gives -2.

We learn from:

(i) That a *positive* number divided by a *positive* number gives a *positive* number.

(ii) That a *negative* number divided by a *positive* number gives a *negative* number, and that a *negative* number divided by a *negative* number gives a *positive* number.

(iii) That a *positive* number divided by a *negative* number gives a *negative* number.

In signs:

$+$ divided by $+ = +$, $\quad -$ divided by $+ = -$,
$-$ divided by $- = +$, $\quad +$ divided by $- = -$.

The rule is seen to be the same as that for multiplication, namely, "*Two like signs give plus; two unlike signs give minus.*"

7. Miscellaneous Examples.

Note.—Except where indicated by brackets, multiplication and division must be done before addition and subtraction.

Example i.—Simplify, $6 + 2 \times -3 - 6 \times -4 \div 7$.

$$6 + 2 \times -3 - 6 \times -4 + 7 = 6 - 6 + 24 + 7.$$
$$= 31.$$

Example ii.—Simplify, $(6 + 2) \times -3 - 6 \times -4 + 7$.

$$(6 + 2) \times -3 - 6 \times -4 + 7 = 8 \times -3 + 24 + 7$$
$$= -24 + 24 + 7$$
$$= 7.$$

Exercise IV (e)

Divide

1. 15 by 3. 2. 15 by -3. 3. -15 by 3. 4. -15 by -3.
5. 3 by 15. 6. 3 by -15. 7. -3 by 15. 8. -3 by -15.

Find

9. $\frac{18}{-3}$. 10. $\frac{-18}{-3}$. 11. $\frac{-3}{-1}$. 12. $\frac{6}{-2}$.

13. $\frac{1}{2}$ of -4. 14. $\frac{0}{4}$. 15. $\frac{-20}{-4}$.

16. $3 \times -2 + 9 - 4 \div -2$. 17. $(2-3) \times -6 - (4-2) \times 3$.

18. $-3 \times 4 - 6 \times -2 + \frac{15}{-3}$. 19. $\frac{8-2}{2} \div 5 \times -2$.

20. $-\frac{4-8}{2} + \frac{10-2}{3+1}$. 21. $(8-6) \times (6-8)$. 22. $(5-9) \times -3$.

23. Verify that $(5 - 3)$ multiplied by $(4 - 7)$ is equal to
$$(5 - 3) \times 4 + (5 - 3) \times -7,$$
and is equal to $5 \times 4 - 3 \times 4 + 5 \times -7 - 3 \times -7$.

24. Simplify (i) $\dfrac{\frac{-6-9}{4}}{\frac{25-20}{-8}}$. (ii) $\dfrac{-\frac{6-9}{4}}{\frac{25-20}{-8}}$.

CHAPTER V

SYMBOLS, COEFFICIENTS, COMMON PROCESSES WITH SYMBOLS

1. In Algebra, numbers are often represented by letters. Draw a straight line AB, of any length, and another CD (fig. 1).

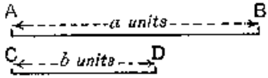


Fig. 1

It is possible to perform certain operations with these straight lines without knowing their length.

Let us represent the number of units of length in AB by the letter a, and the number of units in CD by the letter b; then, if we place AB and CD together in one straight line as in fig. 2, the length of the whole line AD is the sum of a units and b units, which is written $(a + b)$ units.

No simpler answer than $(a + b)$ is possible, since we do not know the values of a and b.

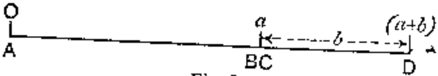


Fig. 2

Of course, if we know a to be 3 in. and b to be 2 in., then we can say that $(a + b)$ is 5 in.

Letters or other characters used to denote numbers are called **symbols.**

2. An arrangement of symbols, such as $a+b$, is called an **expression.**

3. The parts of an expression connected by plus or minus signs are called **terms.** It will be seen later that a term may consist of many symbols.

4. When an expression is used to denote a single measurement like the length of one line, it is a good plan to enclose it in brackets, for the expression acts as one symbol.

Thus the length AD (fig. 2) is $(a + b)$, but it might have been denoted by a single letter, say x.

5. The sum of a and b, then, is $a+b$.

Similarly, the sum of x and y is $x + y$; of x and 2, $x + 2$; and so on.

6. If CD is subtracted from AB, as shown in fig. 3, the number of units of length in AD is $(a - b)$, and this is the simplest statement for the result of subtracting b from a.

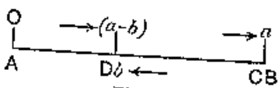


Fig. 3

As before, a simpler result could be obtained if we knew the value of a, and of b.

Thus, b subtracted from $a = a - b$.

Similarly, a subtracted from $b = b - a$,

y subtracted from $x = x - y$,

x subtracted from $y = y - x$,

2 subtracted from $x = x - 2$ and so on.

7. Now suppose that the lines AB and CD happen to be equal in length. Then, if AB measures a units, CD also will measure a units, and on adding AB and CD together, we shall obtain a line AD of $(a+a)$ units (fig. 4).

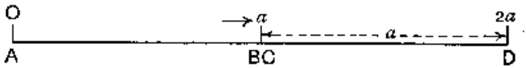


Fig. 4

Now $(a + a)$ is written more simply as $2a$.

If another equal length, DE, is added, then AE is $(a + a + a)$ or $3a$ units.

Similarly, five a's added, i.e. $(a + a + a + a + a)$, are written shortly as $5a$.

8. The number 5 in $5a$ is called the **coefficient** of the term.

The coefficient indicates **the number of the symbols** (or groups of symbols) *added* together.

Note.—The term $5a$ may be taken to mean also 5 times a, although no multiplication sign is placed between the figure and the symbol.

Remember, then, that such a term as $7x$ is a short way of writing seven x's added,

$$\text{i.e. } 7x = x + x + x + x + x + x + x.$$

9. Examples in Addition.

(i) *Add 3a to 5a.*

$5a$ means $a + a + a + a + a$.
$3a$ means $a + a + a$.

Hence $5a + 3a = a + a + a + a + a + a + a + a$,

i.e. eight a's added, which is written $8a$.

$$\therefore 5a + 3a = 8a.$$

Notice that the coefficient of the sum is obtained by adding (algebraically) the coefficients (5 and 3) of the terms.

This applies only to like terms: $5a + 3b$, for example, cannot be simplified in this way.

(ii) *Find the sum of $-3a$ and $5a$.*

$$-3a + 5a = (-3 + 5)a$$
$$= 2a.$$

(iii) *Find the sum of* $-5a$ *and* $3a$.

$$-5a + 3a = (-5 + 3)a$$
$$= -2a.$$

(iv) *Find the sum of* $6a$, $-9b$, $-2a$, $+4b$ *and* $-a$.

Sum of the terms containing $a = 6a - 2a - a = 3a$.
Sum of the terms containing $b = -9b + 4b = -5b$.
Sum of all the terms $= 3a - 5b$.

Exercise V (A)

Simplify the following:

1. $2a + 5a$. 2. $5x + 3x$. 3. $x + 2x$.
4. $3a + 2a + 4a$. 5. $2y + y + 5y$. 6. $2a + 6c + 2c + c$.
7. $6x + 2x - x$. 8. $-6x + 2x + 5$. 9. $5p + 3p$.
10. $5p + (-3p)$. 11. $-5p + (-3p)$. 12. $3a - 2a + 5b - 2b$.

Write down algebraically.

13. $3a$ and $2b$ added. 14. $-5x$ and $2y$ added.
15. $5x$ and $-2y$ added. 16. $5x$ and $-2x$ added.

17. A vessel when empty weighs w gm. If x gm. of water are poured in, what is the total weight?

18. What is the total weight of the solution formed by dissolving x gm. of salt in 100 c.c. of water? (1 c.c. of water weighs 1 gm.)

19. Draw a straight line 1·3 in. long and another 1·6 cm. long. Call the length of the first x units, and that of the second y units. Now draw a line $(3x + 2y)$ units long, and another representing the sum of $-3x$ and $-2y$ units.
Measure all the lines in cm., and check your results.

20. Collect like terms, and so simplify the following:
$$3a + 2b + 4a + 5b + a + b + c + 3b + 5c.$$

21. Simplify
$$3a - 2b - 4a + 5b + a - b - c - 3b + 5c.$$

22. Simplify $3x + 4a - b - x + 2b - 3a + 4b$,
and find its value when $x = 2$, $a = -1$ and $b = 3$.

23. Add the columns of the following example:

$$\begin{array}{r} a + 2b + 3c \\ 2a + b - 2c \\ -5a - 4b + c \\ 3b - 2c \\ 6a - 2b - 3c \\ \hline \\ \hline \end{array}$$

Find the value of each line and of the sum when $a=1$, $b=2$ and $c=3$, and thus check your result. Why would the value $a=0$ not check the sum of the first column?

24. Arrange the following expressions as in Exercise 23, and find their sum:

$3a - 2x + y$, $2x - 3y$, $-2a + 5x - 6y$,
$8a + 3y$, $-5a + x - 5y$, $4a + x + 2y$.

Choose suitable values for these symbols, and check your result.

25. When $x = 2$ and $y = -3$, find the value of each of the following:

(i) $3x + 2y$. (ii) $3x - 2y$. (iii) $2x + 3y$.

(iv) $-2x + 3y$. (v) $\frac{3x}{2} + \frac{2y}{3}$. (vi) $\frac{3x}{2} - \frac{2y}{3}$.

(vii) $3a + 2x - 3y$. (viii) $\frac{x}{3} + \frac{3y}{4} + c$.

26. When $a = 0$, $b = -2$ and $c = 1$, find the value of:

(i) $3a - 2b + 3c$. (ii) $\frac{2a}{3} + \frac{3b}{4} - \frac{3c}{2}$.

27. What do the following become when a, b and c are all equal?

(i) $2a - 3b + c$. (ii) $7a + 2b - 3c$. (iii) $\frac{3a}{5} - b + 2c$.

10. Difference.

From Example i, p. 39, it will be readily understood that when $3a$ is subtracted from $8a$ the result is $5a$,

i.e. $8a - 3a = 5a$.

The coefficient 5 is obtained by subtracting the coefficient, 3, of the term to be subtracted from the coefficient, 8, of the other term. The process may appear simpler when represented as follows:

$$8a - 3a = (8 - 3)a = 5a.$$

Observe that the terms are *like*, i.e. involve the same symbol.

Similarly,
(i) $4a - 7a = -3a$.
(ii) $-4a - 7a = -11a$.
(iii) $-4a - (-7a) = -4a + 7a$
$= 3a$.

11. When subtracting one expression from another, it is sometimes more convenient (though not often) to use the arithmetical arrangement, thus:

From $3a + 2b - 3c$ take $2a - 7b - c + d$.

$$\begin{array}{ll} \text{From} & 3a + 2b - 3c \\ \text{Subtract} & 2a - 7b - c + d \\ \hline & a + 9b - 2c - d \quad \textit{Answer.} \end{array}$$

By the rule for subtraction, page 31, the above example is equivalent to:

$$\begin{array}{ll} \text{To} & 3a + 2b - 3c \\ \text{Add} & -2a + 7b + c - d \\ \hline & a + 9b - 2c - d \quad \textit{Answer.} \end{array}$$

Exercise V (b)

1. (i) From $6a$ take $2a$. (ii) From $2a$ take $6a$.
(iii) From $6a$ take $-2a$. (iv) From $-6a$ take $2a$.
(v) From $-6a$ take $-2a$. (vi) From $-2a$ take $-6a$.
(vii) From $2a$ take $-6a$. (viii) From $-2a$ take $6a$.
Check each result by working the reverse operation.

2. Using the same values for x and y as in No. 19, Ex. V (a), find straight lines to represent:
(i) $(3x - 2y)$. (ii) $(3x - 4y)$. (iii) $(x - 3y)$.
(iv) $(2x - 5y)$. (v) $(-x + 3y)$. (vi) $(-x - y)$.
(vii) $(-2x - (-3y))$. (viii) $0 - (-2y)$.

In each case say whether the result is positive or negative. Check your drawing by measurement and calculation.

3. From $3a - 2b + 7c$ subtract $2a + 5b - c$.

4. (i) From $2x$ subtract 2. (ii) From $2x$ subtract x.
(iii) From 0 subtract x. (iv) From x subtract $-x$.

5. From $2a - 5b + 3$ take $3a + 2b + 5$.

Check your result by substituting 1 for a and 2 for b in each given expression and in the answer.

6. Simplify

$$3a - 2b + 7c + 2a + 5b - c + 3b - 4c - 5a + 3b - 5c - (2a + 2b + c).$$

Find the value when $a = -2$, $b = 3$ and $c = -1$.

7. Find the difference between:

(i) $-x$ and x. (ii) 0 and a.

(iii) 3 and $3x$. (iv) $-x$ and y.

8. An empty crucible weighs x gm. When some copper filings are placed in it, the whole weighs y gm. After heating, the crucible and contents weigh z gm.

Find (i) The weight of copper filings taken.

(ii) The gain in weight after heating.

9. A test-tube and its fittings weigh w gm.

m gm. of pyrolusite and n gm. of chlorate of potash are placed in the tube and the whole heated.

If the weight is then x gm., what is the loss in weight?

If only the chlorate loses weight, what is the loss?

10. A copper ball weighs C gm. in air, W gm. in water and M gm. in turpentine.

Find

(i) The loss of weight in water.

(ii) The loss of weight in turpentine.

(iii) The excess of the loss of weight in water over the loss of weight in turpentine.

12. Product.

We have seen that $3x$ may be taken to mean three times x, or three x's added.

If we were not aware of the number of x's, we could denote it by another symbol, say a, and write ax for the result.

This result ax may be taken to mean also a times x.

Notice that there is no sign between the two symbols.

Similarly, we have:

$x \times y = xy$, $x \times -y = -xy$, $-x \times y = -xy$,

$-x \times -y = xy$, $a \times b = ab$, $a \times -b = -ab$,

$3a \times 2b = 6ab$, $a \times b \times c = abc$, $3a \times b \times 2c = 6abc$.

It should be noted that of the product 6abc, 2, 3, a, b and c are called **factors.**

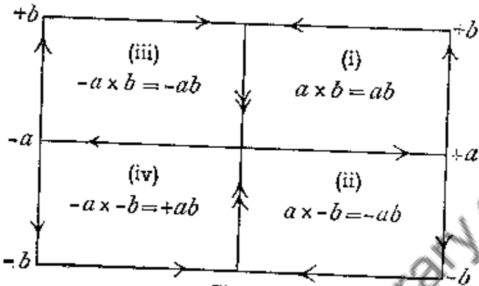


Fig. 5

Fig. 5 represents graphically the products of:
(i) a and b, (ii) a and $-b$, (iii) $-a$ and b, (iv) $-a$ and $-b$.

13. Powers and Indices.

We have seen that: $x \times y = xy$.

Suppose now that x and y happen to be equal; then, if we represent these numbers by straight lines, the lines will be equal, and the rectangle which represents the product will be a square of area x times x (fig. 6).

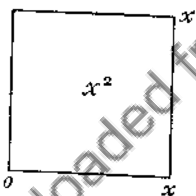


Fig. 6

This product is not written xx, but x^2. (Spoken, x squared.)

The figure **2** is called an **index** (plural indices).

It indicates the number of x's *multiplied* together.

The number x^2 is called also the *second* **power** of x.

Similarly $x^3 = x \times x \times x$ (the third power of x).
$x^6 = x \times x \times x \times x \times x \times x$ (the sixth power of x).

Remembering the rule of signs,

$-x \times x = -x^2$, $\quad -x \times -x = +x^2$,

$-x^2 \times y = -x^2y$, $\quad -x^2 \times -y = +x^2y$,

$-x \times x \times x = -x^3$ (for $-x \times x = -x^2$ and $-x^2 \times x = -x^3$).

14. Products of Powers.

To find $x^3 \times x^2$.

x^3 means three x's multiplied together,

and x^2 means two x's multiplied together;

then, $x^3 \times x^2 = \underbrace{x \times x \times x} \times \underbrace{x \times x}$,

i.e. (three plus two) x's multiplied together, i.e. x^5.

Hence $x^3 \times x^2 = x^{3+2} = x^5$.

N.B.—$x^3 \times x^2$ is not x^6, but x^5.

In words, *the index of the power obtained by multiplying given powers of the same symbol is the* **sum** *of the indices of the given powers.*

EXAMPLES.

$$x^3 \times x^4 = x^{3+4} = x^7,$$
$$x^3 \times -x^4 = -x^{3+4} = -x^7,$$
$$-x^3 \times -x^4 = +x^{3+4} = x^7,$$
$$x^3y \times x^2 = x \times x \times x \times y \times x \times x$$
$$= x \times x \times x \times x \times x \times y$$
$$= x^5y,$$
$$x^2y^3 \times -x^3y = -x^{(2+3)}y^{(3+1)}$$
$$= -x^5y^4,$$
$$(x^2)^3, \text{ i.e. } x^2 \times x^2 \times x^2 = x^{2+2+2} \text{ or } x^{3 \text{ times } 2} = x^6.$$

Note.—When x^2 and x are added, the result is $x^2 + x$. The sum cannot be stated as a single term. Terms which contain the same symbols to the same power are called **like** terms.

EXERCISE V (c)

Complete the following:

1. $x \times a =$ 2. $-x \times -a =$ 3. $ab \times x =$
4. $-3a \times 2x =$ 5. $-\frac{2}{3}x \times 3y =$
6. By means of a figure, show that $(x + 2) \times 3$ equals $3x + 6$.
7. Multiply $x + 2$ by a. 8. Multiply $x + 2$ by -3.
9. Multiply $x + 2$ by $3a$ and by $-3a$.
10. Multiply $3a - 2b$ by $-2x$. Verify your answer by giving numerical values to a, b and x.

11. Taking straight lines 1·3 in. long and 1·6 cm. long respectively to represent x and y, construct figures to represent

(i) $6xy$. (ii) $x(x + y)$. (iii) $x(2x - y)$.
(iv) $2x(3x - 2y)$. (v) $x(x + 2'')$. (vi) $x(3x - 2'')$.

12. Write down answers to the following:

$$a \times a, \quad a \times -a, \quad -a \times a, \quad -a \times -a.$$

13. Show by a drawing that $x \times 3x = 3x^2$, and that

$$x(3x + y) = 3x^2 + xy.$$

14. Write down the answers to the following:

(i) $2x^2y^2 \times -3xy$. (ii) $3ab \times b^2$. (iii) $(2x)^2$.
(iv) $-(2x)^2$. (v) $(-2x)^2$. (vi) $(x^3)^2$.
(vii) $(-3a^2b)^2$. (viii) $(-a)^3$. (ix) $(2a^2)^3$.

Find the value of each of the above when $x = 2$, $y = 3$, $a = -3$ and $b = -1$.

15. The area of a triangle is half that of the rectangle having the same base and altitude (height). If the base is b and the altitude h, the area is $\frac{1}{2}bh$.

Find the areas of triangles having the following dimensions:

Base	Altitude
x	y
$2x$	y
y	$2x$
$3b$	$3h$
b	6

15. Division.

The value of a divided by b is written as $\frac{a}{b}$.

If the terms of the quotient are powers of the same symbol, e.g. $\frac{a^5}{a^3}$, the answer can be simplified by cancelling.

Thus, $$\frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a^2,$$

i.e. $$\frac{a^5}{a^3} \quad \text{or} \quad a^5 \div a^3 = a^{(5-3)} = a^2.$$

Notice that the index of the quotient is obtained by *subtracting* the index of the divisor from the index of the term divided.

(This is of course the reverse of the rule for multiplication, given on p. 45).

Thus, $a^4 \div a^3 = a^{4-3} = a^1 = a.$

Similarly, $\frac{x^7}{x^4} = x^{7-4} = x^3$

and $\frac{-30x^7}{6x^4} = -5x^3$

and $\frac{x^2y^3z}{xy} = xy^2z.$

It should be borne in mind that a product is always exactly divisible by each of its factors, and that the quotient in each case is the product of the other factors.

Exercise V (d)

Simplify:

1. $\frac{-a^2}{a}$. 2. $\frac{-a^2}{-a}$. 3. $\frac{9x^5}{3x^2}$. 4. $-\frac{2x^3}{6x^7}$.

Divide:

5. $-12a^3$ by $4a$.
6. $-3b^5$ by $-2b^2$.
7. $12a^3b^2$ by $4ab$.
8. $-8x^2y$ by $2x$.
9. $3a^2 + 12ab$ by $3a$.
10. $x^2 - xy^2$ by x.

Check each answer by working the reverse operation.

Simplify:

11. $\frac{5xy \times 2ax}{axy}$. 12. $\frac{2ab \times -3ac}{12bc}$. 13. $\frac{x^2 + xy^2}{2x}$.

14. $\frac{x^2y - 3x^2y^3}{xy}$. 15. $\frac{-15a^3b^3 - 3ab^2 + 12ab}{-3ab}$.

16. Add
$$\begin{array}{rrr} 3x^2 & - 2xy & + y^2 \\ -x^2 & + 5xy & - 6y^2 \\ 2x^2 & & - 4y^2 \\ -4x^2 & - 3xy & + 5y^2 \end{array}$$

17. From $3x^2 + 2x + 1$ subtract $2x^2 - 3x + 5$.

18. Multiply $\frac{2}{y}$ by 3.

19. Add $\frac{3}{y}$ and $\frac{2}{y}$. Verify your answer by putting y equal to 4.

20. From $\frac{9}{x}$ subtract $\frac{3}{x}$. Verify your answer.

21. When $x = 2$ and $y = -3$, find the value of:

(i) $\frac{x^2}{y}$. (ii) $\frac{-3x^2}{2y^2}$. (iii) $-\frac{3x^2y^3}{xy}$.

(iv) $\frac{x^2}{y^2} + \frac{y^2}{x^2} - \frac{y}{x}$. (v) $\frac{xy}{x+y}$. (vi) $\frac{6}{x} - \frac{9}{y}$.

22. Divide the numerator and the denominator of $\frac{x^2}{x^5}$ by x^2.

23. Divide the numerator and the denominator of $\frac{ax^3 - bx^3}{ax^3 + bx^3}$ by x^3, and giving numerical values to a, b and x show that the value is not altered.

24. Simplify: $\frac{(8)^2 \times 3}{4} + \frac{(15)^2 \times 4}{5}$.

16. Special cases of Multiplication.

Square and Square Root.

We have seen that $x \times x = x^2$.

This operation is called squaring. Squaring consists of multiplying a number by the same number.

Thus, the square of $-3a$ is $-3a \times -3a = 9a^2$.

17. The reverse operation consists in finding a number whose square is equal to a given number. The result is called a *square root* of the given number.

The sign for the operation is $\sqrt{\ }$.

Any positive number has *two* square roots. For example, the numbers $+3$ and -3 are both square roots of 9, since $3 \times 3 = 9$ and $-3 \times -3 = 9$.

In *arithmetical* work, however, if x is any positive number the symbol $\sqrt{x}$ is always understood to mean the positive square root of x. Thus $\sqrt{9} = 3$, $\sqrt{100} = 10$, $\sqrt{1} = 1$, $\sqrt{2} = 1{\cdot}414..$ On this understanding, the two square roots of 2 are $\sqrt{2} = (1{\cdot}414..)$ and $-\sqrt{2} = (-1{\cdot}414..)$. In *literal* expressions it is customary to use the symbol $\sqrt{\ }$ to denote either of the square roots. Thus $\sqrt{x^2}$ is either $+x$ or $-x$. These roots are often written together in the form $\pm x$. (Spoken, plus or minus x.)

EXAMPLES.

$$\sqrt{9x^2} = \pm 3x, \quad \sqrt{x^6} = \pm x^3, \quad \sqrt{16x^8y^6} = \pm 4x^4y^3.$$

Notice that in finding the *square* of a term, the index is *doubled* and in finding the *square root*, the index is *halved*.

18. Surds.

Surds are roots which cannot be written in terminating form. For example, $\sqrt{8}$ is not quite 3, but 2 decimal something. The decimal part neither recurs nor terminates. However, since $8 = 4 \times 2$ and $\sqrt{4}$ is 2, we can write $\sqrt{8}$ more simply as $2\sqrt{2}$.

(Observe carefully that $2\sqrt{2}$ means twice $\sqrt{2}$ and not $2 + \sqrt{2}$. Contrast this with $2\frac{1}{2}$, which means $2 + \frac{1}{2}$.)

Similarly,
$$\sqrt{18} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2},$$
$$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}.$$

Surds can be treated exactly as algebraic symbols.

19. Logarithms.

Another name for index is Logarithm (abbreviation, log).

Thus in a^3, 3 is the logarithm, and a is called the **base.**

These facts are usually expressed as follows:

The logarithm of a^3 to the base a is 3.

The statement means that 3 is the index of the power to which the base a must be raised to give the answer a^3.

$10^3 = 1000$, therefore the log of 1000 to the base 10 is 3,
$10^5 = 100000$, „ „ 100000 „ 10 is 5.

The short way of writing these is
$$\log_{10} 1000 = 3,$$
$$\log_{10} 100000 = 5.$$
Similarly,
$$\log_a(a^3) = 3.$$

Notice that the base is written to the right of, lower down and smaller than the abbreviation log.

Exercise V (e)

1. Find the square of:

1, -1, $2x$, $-3x$, $-5x^2$, $3x^3$, $4\sqrt{x}$, $\sqrt{-x}$, $\sqrt{x^3}$, $3(a + b)$.

2. Find the square roots of the following. Verify your answers by squaring:

$25a^2$, $9b^4$, $49b^6$, $64y^{10}$, $25(a + b)^2$.

3. Why cannot you find the square root of -9?

4. Simplify:

$$\sqrt{\frac{x^2}{4y^4}}, \quad \frac{\sqrt{x^2}}{4y^4}, \quad \sqrt{\frac{1}{x^2}}, \quad \frac{1}{\sqrt{x^2}}, \quad \sqrt{x^2 + 8x^2}.$$

5. Write as simply as possible:

$\sqrt{16}$, $\sqrt{25}$, $\sqrt{64}$, $\sqrt{100}$, $\sqrt{144}$, $\sqrt{12}$, $\sqrt{20}$, $\sqrt{28}$, $\sqrt{72}$, $\sqrt{32}$, $-\sqrt{75}$, $3\sqrt{16}$, $3\sqrt{32}$, $x\sqrt{8}$, $\sqrt{8x^2}$, $\sqrt{32x^4y^2}$, $\sqrt{x^2+x^2}$.

6. Show that $\sqrt{4 \times 9}$ is equal to $\sqrt{4} \times \sqrt{9}$, but that $\sqrt{4+9}$ is not equal to $\sqrt{4} + \sqrt{9}$, and that therefore $\sqrt{4} + \sqrt{9}$ is not equal to $\sqrt{13}$.

7. Write the following as powers of 10, and state the logarithm of each to the base 10:

1000, 100, 1, 10,000, 1,000,000 10 million.

8. Find the following:

$\log_5 25$, $\log_2 32$, $\log_2 64$, $\log_4 64$, $\log_4 \frac{1}{8}$.

9. Write down the product of the following in the form of powers:

2^2 and 2^3, 2 and 2^4, 3^2 and 3^3, 5^4 and 25^3.

10. What is $\log_3(3^2 \times 3^3)$?

Logarithms are more fully dealt with in Chap. XVII, which may be read at once if desired.

20. Representation by Graph.

The numerical value of, say, $4a$ depends, of course, upon the value of a. Thus, if a is 1, then $4a$ is 4; if a is 2, $4a$ is 8, and so on. The value of $4a$ changes then with the value of a. A change in the value of a from 1 to 2 makes a change in the value of $4a$ from 4 to 8.

There is a very convenient way of representing values which are subject to change, namely, by means of graphs.

You have probably met with it in your lessons in Arithmetic or Geography. Occasionally you meet with it in newspapers.

The following are examples:

(i) Fig. 7 shows the amount of cotton imported by the United Kingdom during the years 1903 to 1912.

The years are marked on the horizontal line, and the length of each vertical line represents to scale the quantity of cotton imported during the year stated at the foot of the line. Thus, in 1906, the quantity was 20 million centals, i.e. 2000 million pounds.

The advantages of graphical representation are:

(1) At a glance, the amounts for each year can be compared.
(2) The diagram is more striking than a list of numbers.

(3) The lowest and the highest quantities are readily picked out.

(4) Some idea as to the importance of the changes (increases or decreases) can be quickly gained.

(5) An opinion can be readily formed as to whether there is, on the whole, an upward or a downward tendency.

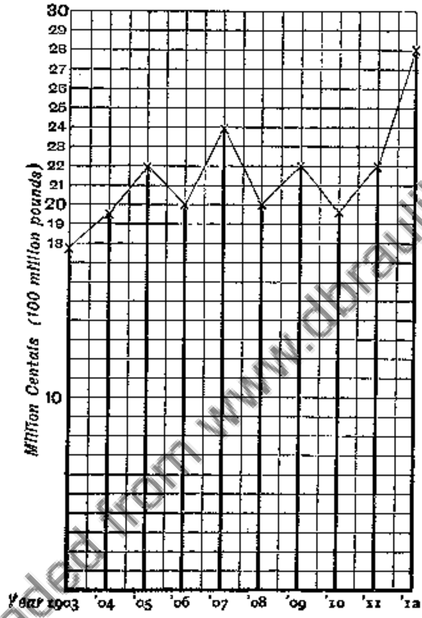


Fig. 7

(6) Exceptional cases stand out prominently.

(7) A close approximation to the average value is quickly made.

If the heads of the lines be joined by straight lines, rise and fall are strikingly distinguished.

(ii) Height of the tide and the date.

Fig. 8 shows the height of successive tides and the date.

It is interesting to compare the graphs for several months, and also to see whether the changes agree with the phases of the moon. The wave form of this graph is apparent.

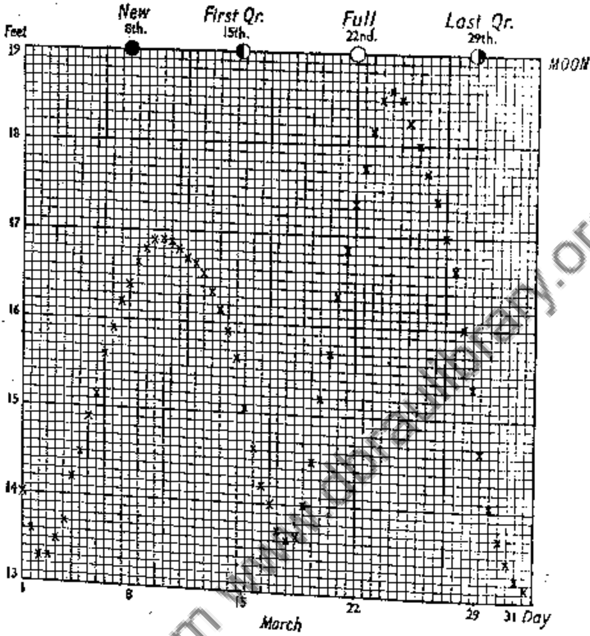


Fig. 8

(iii) Depth of a swimming bath and the length.

Fig. 9 shows the depth of a swimming bath at places along its length.

There is a question concerning these graphs which is of the utmost importance: namely, under what conditions can a graph be used to determine values which are not definitely shown by marked points? Refer to the graph giving the depths of a swimming bath at various distances along its length.

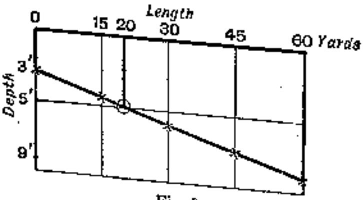


Fig. 9

The points showing the depth appear to follow a straight line. Draw the straight line, and drop the "depth line" from the point denoting 20 yd., and read off what the depth appears to be at this place. Now this result is probably correct, but as to whether it is absolutely correct depends upon whether the change in depth, which we have found by measuring the depth at certain

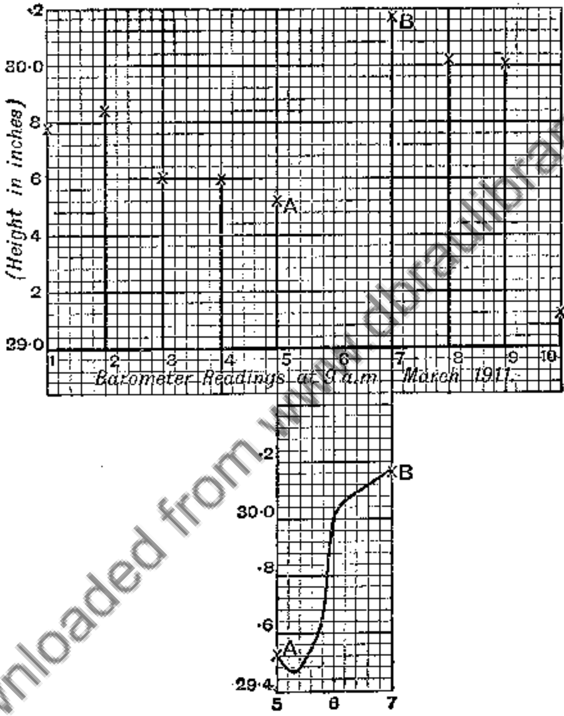


Fig. 10

places, is uniform. The more measurements we make the better judgment we can form on this matter, and, of course, the matter could be decided definitely by examining the bath when empty. The correctness of the answer, found from the diagram, depends upon whether the law suggested by the change in depth found at certain places is followed throughout the whole length of the bath.

(iv) Turning now to the diagram of the readings of the barometer (fig. 10), it is observed that there appears to be no uniformity in the sequence of the points. They do not lie on a smooth curve even, like those of diagram 12.

You will notice that the reading for March 6th is missing. Can we, from the other readings, determine the missing reading? The behaviour of the barometer is so erratic that this is impossible. Joining A and B, the neighbouring points, by a straight line will not help us.

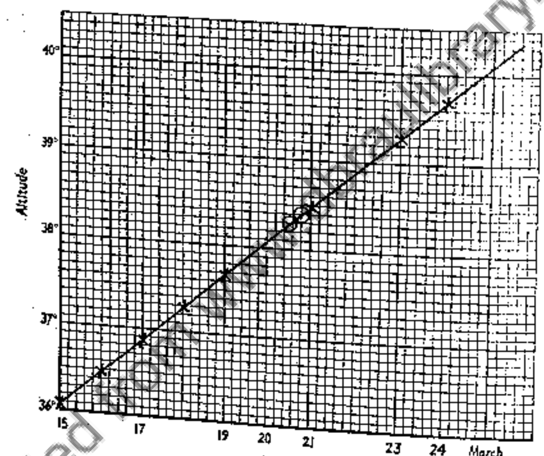


Fig. 11

To see how difficult the problem is, it is only necessary to examine the curve of one of those instruments which, by means of a pen and clockwork, automatically draw a continuous graph for a day or a week. Such a curve is shown in the lower part of fig. 10. In it the missing reading will be found.

These various changes appear to follow no law. If there is a law, it is a very complex one.

Even where the points lie on a smooth curve as in fig. 12, or a straight line as in fig. 11, which suggests that some law exists, we must not conclude that the graph will give us values for intervals between the points for which it is constructed. E.g. referring to fig. 11, the graph of the sun's altitude at noon, if we draw the line midway between the two noons, March 20 and 21, that is, at a point which would represent midnight, the altitude obtained is 38·3°, which is, of course, ridiculous at the latitude of observation.

Again, at a point which would represent 6 a.m., the reading is 38·4°, which also is out of the question, since the sun at this time is just rising.

The changes for a part of the day between sunrise and sunset, March 21st, are shown in fig. 12.

On the other hand, this figure can be used for values between the times shown, for the changes are continuous between the times of sunrise and sunset.

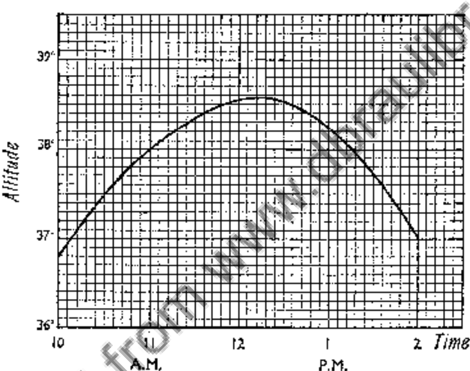


Fig. 12

21. We can represent the various values of $4a$ in a similar manner.

Represent the values of a along the horizontal line, and the corresponding values of $4a$ along lines at right angles (fig. 13).

If squared paper is used, the lines are already drawn.

Arrange the values of a and of $4a$ in a table, thus:

a =	1	2	3	4	etc.
$4a$ =	4	8	12	16	etc.

Mark the ends of the lines representing $4a$ by a cross (×).

Now, what line do these end points follow?

Draw the line.

See if this line enables you to find values of $4a$ not already calculated from values of a.

Try, for example, values of a beyond 10, or between 4 and 5.

Determine whether the line will hold for negative values of a.

State your conclusions clearly.

22. When two things are so associated or interdependent that one becomes definite when the other is made definite, the first is said to be a **function** of the second.

Thus the pressure of the atmosphere is a function of time.

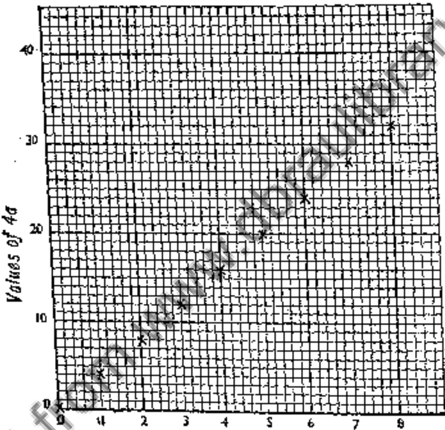


Fig. 13

The weight of water is a function of the volume. The value of gold is a function of the weight, and so on.

In many cases the real connexion between the things, or the law connecting them, is unknown. There are, however, some functions which can be expressed simply, and we shall deal with some of them later.

The value of $4a$ is a function of a.

When a definite value is given to a, the value of $4a$ is determined.

The graph gives correct values of $4a$ when a is positive, when a is zero and when a is negative. It is true when a is numerically great and when a is numerically small. Moreover, and this is important, we can find a value for a which will give any value we please for $4a$, i.e. it is possible for $4a$ to have any value whatever, and there will be a corresponding value for a also.

EXERCISE V (F)

1. Examine carefully the graphs in figs. 7 to 13, and write down the conclusions you draw from each.
2. Make marks, say, a centimetre apart, on a wax candle. Screen the candle from draught, light it, and note the time at which the marks disappear.

   Represent time on a horizontal line, and the length of the candle remaining at the noted time by lines at right angles to the time line.

   After four or five observations, predict how long the candle will last.

   Examine the graph, and draw your conclusions.
3. Fix a burette vertically in a stand. Fill the burette with coloured water, and turn the tap so that the liquid runs out slowly.

   Using a seconds' watch, take the time at which the surface of the liquid in the burette passes the 0, 5, 10, 15, etc., graduation marks.

   Construct a graph of your observations, examine it, and draw conclusions.
4. Take a steel spiral spring or a piece of rubber cord. Suspend it from a rigid bracket or nail, and to the lower end attach a light pan. In order to measure the length of the spring or cord, erect a ruler by its side. Increase gradually the weight applied, and note the length for each pull.

   Construct a graph of your results, and draw conclusions.

CHAPTER VI

BRACKETS. EASY FRACTIONS

1. Brackets.

We have seen on p. 38 that when we wish terms, added or subtracted, to be considered together and not apart, they are bracketed.

Thus, $(3x - 2y)$ may be considered as one term, like the symbol a.

The bracket may have a coefficient, e.g.

$$5(3x - 2y) \quad \text{or} \quad -4(3x - 2y).$$

As a matter of fact, $(3x - 2y)$ has the coefficient 1, which it is not necessary to write.

Now the question is, suppose we wish to remove the bracket and separate the terms, what change will a coefficient make?

The meaning of $5(3x - 2y)$ may be taken to be $(3x - 2y)$ multiplied by 5, i.e. $3x \times 5 - 2y \times 5$ or $15x - 10y$; or, we may take it as meaning five such expressions added. Thus:

$$\begin{array}{r} 3x - 2y \\ 3x - 2y \\ 3x - 2y \\ 3x - 2y \\ 3x - 2y \\ \hline 15x - 10y \end{array}$$

The result is the same as before. The first is, however, the simpler operation.

Special cases:

(i) $(3x - 2y)$. The coefficient here is 1, and multiplying the terms by it does not alter them. Hence, $(3x - 2y) = 3x - 2y$, without the bracket.

(ii) $-(3x - 2y)$. The coefficient here is -1, and the effect of multiplying by -1 is to change the signs. Thus: $-(3x - 2y) = -3x + 2y$, without the bracket.

(iii) $-4(3x + 2y) = -12x - 8y$.

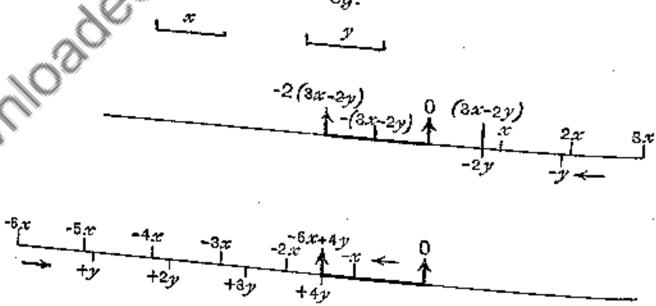


Fig. 1

To show graphically that $-2(3x - 2y) = -6x + 4y$.

Let x and y be represented by the lines shown (fig. 1).

The upper long line shows $(3x - 2y)$ on the right, and $-2(3x - 2y)$ on the left of the zero.

The lower long line shows on the left of the zero $-6x + 4y$.

It is seen that the two results $-2(3x - 2y)$ and $-6x + 4y$ are alike in magnitude and sign.

Note.—Had $(3x - 2y)$ been to the left of the zero, $-2(3x - 2y)$ would have been obtained by taking twice this distance on the right side of the zero.

2. Insertion of Brackets.

In the reverse operation, to obtain the terms to be placed inside the bracket, divide the given terms by the number it is proposed to place outside. Thus:

EXAMPLE.—To bracket $-4x + 6y$.

The terms can be divided by -2, giving $-2(2x - 3y)$.

Incidentally, we have obtained the factors of $-4x + 6y$, for -2 and $(2x - 3y)$ when multiplied together give $-4x + 6y$, and each is simpler than the product.

Note.—In an expression such as $\frac{2x - y}{2}$, the line separating the numerator from the denominator acts like a bracket. Not only $2x$, but $-y$ also, has to be divided by 2.

$\frac{2x - y}{2}$ may be written in the form $\frac{1}{2}(2x - y)$.

If the line or link be removed, the result is $\frac{2x}{2} - \frac{y}{2}$ or $x - \frac{y}{2}$.

The following case is very important.

Show that $(a - b) = -(b - a)$.

$$(a - b) = a - b \quad \text{(removing the bracket)}$$
$$= -b + a \quad \text{(rearranging the terms)}$$
$$= -(b - a) \quad \text{(bracketing, taking out the common factor } -1\text{)}.$$

EXERCISE VI (A)

Show graphically that:

1. $2(2x - 3y) = 4x - 6y$.
2. $-3(x - 2y) = -3x + 6y$.
3. $-(x - y) = -x + y$.
4. $-(x + y) = -x - y$.

5. Five bags each containing 30 sovereigns and a bill showing a debt of £8 are emptied out. State the total contents.

Verify your answer to each of the following exercises, by giving numerical values to the symbols.

Remove the brackets and simplify when possible.

6. $5(2x - 3y)$.
7. $-5(2x - 3y)$.
8. $x(2x - 3y)$.
9. $-x(2x - 3y)$.
10. $-y(2x + 3y)$.
11. $-y(2x - 3y)$.
12. $a(a + b - c) + b(b + c - a) + c(a + c - b)$.
13. $3a - 2(4b - 5c) + 3b - (4c - 3a) - 3c - (4a + 3b)$.
14. $3(x - 2y) - 2(3x + 4y) + x(3x - 2y)$.
15. $(2a + 3b) - 2(3b - 5a)$.
16. $2(3a + 2b) - 3(a - 3b) - 7(a + b)$.
17. $-(a - b) - (b - c) - (c - a)$.

Bracket the following; check your answers by removing the brackets.

18. $2x + 2y$.
19. $-2x - 2y$.
20. $-2x + 6y$.
21. $3a - 6b + 12c$.
22. $3a - 6b + 5c + 25d$.
23. $3a - 6b - 5c + 25d$.
24. $a^2 - ab - c^2 + cd$.
25. $a^2 + ab - ab + b^2$.
26. $x^3 - 3x^2y - xy^2$.
27. Show that $-(b - a) = (a - b)$.

Simplify:

28. $\dfrac{3x - 6y}{3}$.
29. $\dfrac{15x^2 + 20y^2}{5}$.
30. $\dfrac{10y - 4x}{-2}$.

3. Complex Brackets.

It is sometimes found necessary to bracket terms which form part of an expression already bracketed.

The kinds of brackets used are:

(i) Square [ ].
(ii) Braces { }.
(iii) Plain ( ).
(iv) Vinculum, line or link $\overline{\quad\quad}$.

An example will show the use of these brackets.

$$-2[6a + b + 3\{8b - 4(3 - \overline{2a - b})\}].$$

You will notice that the innermost bracket is the vinculum, and the outermost the square bracket.

To simplify this expression, it is usual to commence with the innermost and to remove the brackets step by step.

Thus:

(i) $-2[6a + b + 3\{8b - 4(3 - 2a + b)\}]$. Vinculum removed.

(ii) $-2[6a + b + 3\{8b - 12 + 8a - 4b\}]$. Plain brackets removed.

(iii) $-2[6a + b + 24b - 36 + 24a - 12b]$. Braces removed.

(iv) $-12a - 2b - 48b + 72 - 48a + 24b$. Square brackets removed.

(v) $-60a - 26b + 72$. Terms collected.

4. As an exercise in the reverse process, the pupil should try to re-insert the brackets in the foregoing example, commencing from the last line but one.

Insert the square bracket first, and the others in successive steps.

Exercise VI (b)

Simplify:

1. $x - 2(x + 3y) - 3\{y + 2(2x - y)\}$.
2. $4b - [a - \{2a(x + y) - 3a(x - y)\}]$.
3. $6a - [2a - \{3a - (5a + 3)\}]$.
4. $12x - 2[x - 3\{x - 4(x - \overline{2 - x})\} + x]$.
5. $2[10 - \{8 + (3 + 6) - 2\} + 5] + 3$.
6. $2x - [3y - \{x + 2(3y + 1)\} + 2\{2y - 3(2x - 1)\} + 2x]$.

Insert brackets where possible in the following:

7. $6a^2 - 2a^2b + 2ab^2$.
8. $a^2x^2 + axy - a^2y^2$.
9. $ab + ac - abc$.
10. $ax^2 + bx - cy^2 + dy$.
11. $a(x + y) + ac$.
12. $(a + b)(x + a) - (a + b)(x + b)$.
13. $(p + q)(x + y) + (p + q)c$.
14. $(p + q)(x + y) + (p + q)(y + z)$.
15. $(p + q)(x + y) - (p + q)(x - y)$.
16. $a^2 + ab + ab + b^2$.
17. The long and short sides of a rectangle or oblong are respectively a and b units in length. Represent the perimeter in as simple a form as possible.

    Find the value of your result when $a = 3{\cdot}6$ in. and $b = 2{\cdot}5$ in.

18. The length, breadth and height of a rectangular room are respectively a, b and h. Find an expression for the total area of the walls, floor and ceiling.

19.
$$\frac{\pi h a^2}{3} + \frac{\pi h a b}{6} + \frac{\pi h b^2}{3}.$$

Find the value when $h = 6{\cdot}32$, $a = 4$, $b = 3$, and $\pi = 3{\cdot}142$.

20. Given that $\sqrt{3} = 1{\cdot}732...$, evaluate in the shortest way
$$8\sqrt{3} + 2{\cdot}56\sqrt{3} - 7\sqrt{3} - 1{\cdot}36\sqrt{3}.$$

5. Factors of Simple Terms.

The factors of a term or expression are simpler terms or expressions which when multiplied together produce the term or expression.

The simplest factors are those which cannot be split up into simpler factors. *In Arithmetic, such factors are called Prime Factors.*

The factors of a^2bx are a, a, b, x, because $a \times a \times b \times x$ equals a^2bx, and each is simpler than a^2bx.

The prime factors of 30 are $2 \times 3 \times 5$, because $2 \times 3 \times 5 = 30$, and none of the numbers 2, 3 and 5 can be split into simpler factors.

The factors of $-36a^2b^3$ are $-2, 2, 3, 3, a, a, b, b, b$, because
$$-2 \times 2 \times 3 \times 3 \times a \times a \times b \times b \times b = -36a^2b^3.$$

It is unnecessary to write the same factors at such length, the following shorter form is quite good: $-(2)^2, (3)^2, a^2, b^3$.

6. Highest Common Factor and Lowest Common Multiple.

It is an easy matter to determine the Highest Common Factor (H.C.F.) in a set of algebraic terms.

Thus, in the terms a^3b^2c, $a^2b^3c^2$, $-a^4bc^3d^2$, it is readily seen that the highest power of a which is a factor of all three terms is a^2, that the highest power of b which is a factor of all the terms is b, and that the highest power of c, common to all three terms, is c; also that d is not a common factor, since d to the first or any higher power appears in neither the first nor the second term.

The H.C.F. is therefore a^2bc.

It will be noticed that the H.C.F. contains the lowest power of each symbol contained in all the terms. *E.g.* a is contained in all the terms, the lowest power being a^2, which appears in the second term.

The result can be checked by dividing each term by the H.C.F. The quotients should have no common factor other than unity.

$$\frac{a^3b^2c}{a^2bc} = ab,$$
$$\frac{a^2b^3c^2}{a^2bc} = b^2c,$$
$$\frac{-a^4bc^3d^2}{a^2bc} = -a^2c^2d^2.$$

Neither a, b, c nor d is contained in all three quotients.

If there are coefficients, of course the H.C.F. of these also must be found.

Thus the H.C.F. of $16a^3b^2c$, $24a^2b^3c^2$, $-20a^4bc^3d^2$ is $4a^2bc$.

7. The determination of the Lowest Common Multiple (L.C.M.) of a number of algebraic terms is equally simple.

EXAMPLE.—Find the L.C.M. of

$$a^3b^2c, \quad a^2b^3c^2 \quad \text{and} \quad -a^4bc^3d^2.$$

The answer must contain every symbol in the terms; but the result, to be the *lowest* multiple, must not contain any power of a symbol higher than the highest appearing in the terms. Thus, a^4 in the third term is the highest power of a in all three terms. Hence the L.C.M. must of necessity contain a^4, but no higher power; for the same reason the L.C.M. must contain b^3, c^3 and d^2. The L.C.M. is therefore $a^4b^3c^3d^2$.

Any L.C.M. can be checked by dividing it by each of the terms. The quotients should have no common factor other than unity.

$$\frac{a^4b^3c^3d^2}{a^3b^2c} = abc^2d^2,$$
$$\frac{a^4b^3c^3d^2}{a^2b^3c^2} = a^2cd^2,$$
$$\frac{a^4b^3c^3d^2}{-a^4bc^3d^2} = -b^2.$$

Neither a, b, c nor d appears in all three quotients.

If there are coefficients, the L.C.M. of the numbers also must be found. Thus, the L.C.M. of

$$16a^3b^2c, \quad 24a^2b^3c^2, \quad -20a^4bc^3d^2 \quad \text{is} \quad 240a^4b^3c^3d^2.$$

Note.—In H.C.F. and L.C.M. the result may be either positive or negative. It is usual, however, to give the positive value only.

8. Easy Fractions.

The four processes, addition, subtraction, multiplication and division, are carried out in Algebra in the same way as in Arithmetic.

Addition and Subtraction

EXAMPLE i.—Find

$$\frac{3}{2bc} - \frac{1}{ac} + \frac{2}{3ab}$$
$$= \frac{9a - 6b + 4c}{\text{L.C.M., } 6abc}.$$

Find the L.C.M. of the denominators.

Reduce all the fractions to this common denominator, e.g. in the case of $\frac{3}{2bc}$, divide $2bc$ into $6abc$ and multiply the numerator 3 by the quotient, $3a$.

EXAMPLE ii.

$$\frac{a + 2b}{3c} - \frac{b - 3c}{2a} - \frac{4ab - 3bc}{6ac}$$
$$= \frac{2a(a + 2b) - 3c(b - 3c) - (4ab - 3bc)}{\text{L.C.M., } 6ac}$$
$$= \frac{2a^2 + 4ab - 3bc + 9c^2 - 4ab + 3bc}{6ac}$$
$$= \frac{2a^2 + 9c^2}{6ac}.$$

The line under each numerator acts as a vinculum.

Note the plain brackets.

Notice the change of signs in some cases.

Multiplication and Division

EXAMPLE iii. $\frac{a^2b}{cd^2} \times \frac{a}{c} = \frac{a^2b \times a}{cd^2 \times c} = \frac{a^3b}{c^2d^2}.$

EXAMPLE iv. $\frac{2a^2b}{3cd^2} \div \frac{4a}{9c} = \frac{2a^2b}{4a} \times \frac{9c}{3cd^2} = \frac{ab}{2} \times \frac{3}{d^2} = \frac{3ab}{2d^2}.$

Observe that cancelling, that is, the dividing of a numerator and a denominator by a common factor, is carried out wherever possible.

EXERCISE VI (c)

Find the H.C.F. and L.C.M. of:

1. x^2y^2 and x^3y. 2. $a^2b^3c^4$ and ab^2c^3. 3. a^4 and a^3b^3.

4. $10x^4y^3$ and $15x^3y^4$. 5. $3ab^2c^3$ and $6a^2bc^2$.

6. $a^4b^5c^3$, $a^3b^4c^5$ and $a^5b^3c^4$. 7. $2bc$, ac, $3ab$.

8. $3c$, $2a$, $6ac$. 9. 25, $5ab$ and $10bc$.

Simplify:

10. $\dfrac{ax + bx}{x}$. 11. $\dfrac{10a + 6ad}{2a}$. 12. $\dfrac{3x + 3y}{2x + 2y}$.

13. $\dfrac{a}{x} + \dfrac{2}{x}$. 14. $\dfrac{a}{x} + \dfrac{b}{2x}$. 15. $\dfrac{x}{a} + \dfrac{a}{x}$.

16. $\dfrac{a - b}{2b - 2a}$. 17. $\dfrac{x + y}{a - b} - \dfrac{2x - y}{2a - 2b}$. 18. $\dfrac{a}{b} \times \dfrac{b}{c}$.

19. $\dfrac{ab}{cd} \div -\dfrac{a}{d}$. 20. $\dfrac{3x}{4y} + \dfrac{2y}{3x} - \dfrac{5x}{12y}$.

21. $\dfrac{x}{2y} + \dfrac{2y}{3x}$. 22. $\dfrac{5a + 1}{6} - \dfrac{a + 1}{2} - \dfrac{3a + 1}{4}$.

23. $\dfrac{2y - 3}{3} - \dfrac{3y + 5}{5} + \dfrac{5y + 3}{6} - \dfrac{7x + 5}{10}$.

24. $\dfrac{ax + ay}{3bx} \div \dfrac{3x + 3y}{6ax}$. 25. $\dfrac{\dfrac{3ax - 6bx}{7a^2b^2} \times \dfrac{3ab}{6xy}}{\dfrac{ay - 2by}{14bx}}$.

26. $(5\sqrt{3} - 2\sqrt{3}) \times (4\sqrt{3} + \sqrt{3})$.

27. $\dfrac{2ax + 2}{2ax} \times \dfrac{3bx}{3bx + 3}$. 28. $\dfrac{ab}{ab + b^2} \div \dfrac{b}{a + b}$.

29. Divide $ab(ab + b^2)$ by b. 30. Multiply $3b(a + 1)$ by $2a$.

CHAPTER VII

GEOMETRY

Formal Reasoning.

A student's comprehension of a proof in mathematics, especially in Geometry, depends on his ability to draw correct conclusions from given data.

The following exercises are axiomatic, that is, so elementary as to need no proof. They occur over and over again in the subject.

The symbols a, b, c, etc., may be taken to represent, say, the length of lines or the size of angles.

Fig. 1

Fig. 2

1. If $a = b$ and $b = c$, compare a and c (fig. 1).
   The axiom is, "Things equal to the same thing are equal to one another."
2. If $a = b$ and $c = d$, to what is $(a + c)$ equal? (fig. 2).
   The axiom is, "If equals be added to equals the wholes are equal."
3. If $a = b$ and $c = d$, to what is $(a - c)$ equal?
   The axiom is, "If equals be taken from equals the remainders are equal."
4. If $a + b = b + c$, what conclusion do you draw?
5. If $a = b$ and $b > c$, compare a and c.
6. If $a > b$ and $b > c$, compare a and c.

Other axioms are:

The whole is greater than its part and is equal to the sum of its parts.

The same fractions of equal things are equal, e.g. halves of equal things are equal.

The same multiples of equal things are equal.

FUNDAMENTAL PROPERTIES OF TRIANGLES

PROPOSITION I

Any two sides of a triangle are together greater than the third side.

This is almost self-evident, especially if it is conceded that the straight line is the shortest distance between two points.

PROPOSITION II

The sum of the three angles of a triangle is one straight angle.*

Consider $\triangle ABC$ (fig. 3).

Take a strip of transparent paper and on it mark an arrow to indicate direction. Use a pin as a pivot.

Pivot the arrow at A so that it points in the direction AB.

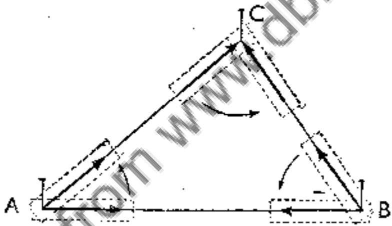


Fig. 3

Rotate the arrow through angle A. It now points along AC. Push it along AC without changing its direction until the arrow head reaches C.

Pivot the head at C and rotate the arrow through angle C. It now points in the direction BC.

Draw the arrow down to B, pivot it at B and rotate it through the remaining angle B.

The arrow now points from B to A, that is, in the opposite direction to which it originally pointed. In rotating through the three angles of the triangle the arrow has turned through a straight angle or 180°.

Observe that all the rotations were of the same kind, in this case anticlockwise.

* Discovery attributed to Pythagoras (569–500 B.C.).

Additional Exercises.

Take a piece of transparent paper and add the three angle a triangle together by tracing them so that they lie side by s and have the same point as vertex. The transparent paper placed so that this point is at each corner in turn.

Letter the angles on the transparent paper according to t angles of the triangle. The angles together form a straight ang

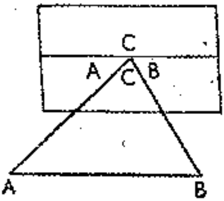


Fig. 4

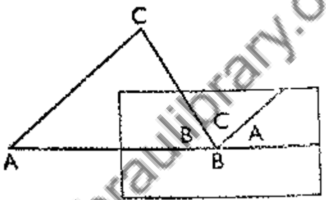


Fig. 5

Place the transparent paper first as in fig. 4, then as in fig. 5, and observe that in fig. 4 you have a straight line through C parallel to AB, and in fig. 5 a straight line through B paralle to AC, and also that the exterior angle formed by producing AB is equal to the sum of the two opposite interior angles at A and C

Special Cases.

(*a*) *If a triangle has three equal angles each is* 60°.

(*b*) *If one angle of a triangle is a right angle then the other tw angles together equal a right angle.*

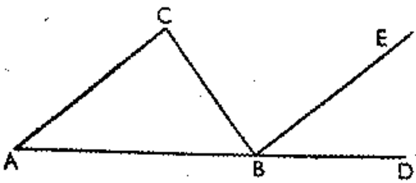


Fig. 6

Now see if you can follow this formal proof.

Given: Any △ABC.

Prove: *The sum of its angles is a straight angle.*

Proof: Produce one side, say AB, to D (fig. 6), and throu B draw straight line BE parallel to AC.

Then since BE is parallel to AC, and CB is a transversal,

$$\angle CBE = \text{alt. } \angle ACB,$$

and since ABD is a transversal,

$$\text{ext. } \angle EBD = \text{opp. int. } \angle CAB.$$

But $\angle ABC$, $\angle CBE$ and $\angle EBD$ together make a straight angle, namely, ABD.

$\therefore$ $\angle ABC$, $\angle ACB$ and $\angle CAB$ together equal a straight angle.

Q.E.D.*

Exercise VII (a)

1. Perform the rotation exercise (fig. 3) in a clockwise direction starting at B.
2. Construct a triangle having its three angles equal, and a triangle with one of its angles a right angle.
3. Two angles of a triangle are respectively 50° and 70°. What is the remaining angle?
4. One angle of a triangle is 72° and the other two are equal. What is the size of each of the equal angles?
5. The angles of a triangle are repectively $x°$, $2x°$ and $3x°$. Find them.

PROPOSITION III

The exterior angle formed by producing a side of a triangle is equal to the sum of the opposite interior angles.

Referring to the previous proof and fig. 6,

since $\angle CBE = \angle ACB$

and $\angle EBD = \angle CAB,$

by addition,

$$\angle CBE + \angle EBD = \angle ACB + \angle CAB.$$

But $\angle CBE + \angle EBD = \text{ext. } \angle CBD.$

$\therefore$ ext. $\angle CBD$ = opp. int. $\angle ACB$ + opp. int. $\angle CAB$.

Q.E.D.

Question.—How does the exterior angle formed by producing a side of a triangle compare with either of the opposite interior angles?

* When a proof is concluded, it is usual to put the letters Q.E.D. at the end. They are the initial letters of the Latin words, "Quod erat demonstrandum," meaning, "which was to be proved". They indicate that the proof is finished.

EQUALITY OR CONGRUENCE OF TRIANGLES

PROPOSITION IV

A triangle has seven elements, namely, three sides, three angles, and area.

Two triangles are **equal in all respects,** i.e. sides to sides, each to each, angles to angles, each to each, and area to area, *if one of the following sets of conditions is true:*

(a) *If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other.*

(b) *If two angles and a side of one triangle are respectively equal to two angles and the corresponding* * *side of the other.*

(It follows from II (p. 67) that the third angles of the triangles also are equal).

(c) *If the three sides of one triangle are respectively equal to the three sides of the other.*

These sets of conditions are very important and must be remembered.

Note.—Triangles equal in all respects are often called **congruent triangles.**

Proofs of the Congruence of Triangles by Superposition.

CONDITION (*a*).—**If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, the triangles are equal in all respects.**

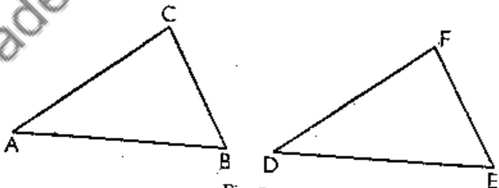


Fig. 7

Preliminary.—Place a piece of transparent paper over $\triangle$DEF (fig. 7). Mark in pencil as accurately as possible the sides DE and DF, being careful to see that the included angle EDF is accurately copied, and that the points E and F are correctly marked.

* Corresponding sides are opposite equal angles.

Remove the paper and join the ends of the lines. Letter the triangle ABC and test your work by seeing if your triangle ABC fits $\triangle$DEF.

Given: $\triangle$ABC and $\triangle$ DEF with

$$AB = DE$$
$$AC = DF$$
$$\text{includ. } \angle BAC = \text{includ. } \angle EDF.$$

Prove:

$$BC = EF$$
$$\angle ABC = \angle DEF$$
$$\angle ACB = \angle DFE$$
$$\triangle ABC = \triangle DEF \text{ in area.}$$

Proof: Place $\triangle$ABC on $\triangle$DEF so that A falls on D and AB lies along DE.

Then AC will lie along DF, because $\angle BAC = \angle EDF$,

and B will fall on E, because AB = DE,

and C will fall on F, because AC = DF,

and since BC and EF are straight lines fitting at both ends, they entirely coincide, and are therefore equal.

Since all the boundaries coincide,

$\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$, and $\triangle ABC = \triangle DEF$ in area.

Q.E.D.

Exercise VII (b)

1. Turn the transparent paper over and notice that the triangles are still equal, in all respects, although not similarly placed. You may meet such triangles.
2. Construct the triangle two sides of which are 2 in. and 3 in. respectively, and include the angle 60°.
   Measure the remaining side and angles.

Condition (*b*).—**If two angles and a side of one triangle are respectively equal to two angles and the corresponding side of another triangle, the triangles are equal in all respects.**

Preliminary.—Trace on transparent paper side DE and angles EDF and DEF of the $\triangle$DEF (fig. 8), making the lines DF and EF so short that they have to be produced to meet. Remove the transparent paper and using a ruler produce the lines to meet. Letter the triangle ABC, and see if it fits $\triangle DEF$.

Given: $\triangle ABC$ and $\triangle DEF$ with

$$\angle BAC = \angle EDF$$
$$\angle ABC = \angle DEF$$
$$AB = DE$$

(fig. 8).

Prove: $\triangle ABC = \triangle DEF$ in all respects.

Proof: Since the angles of any triangle together equal a straight angle it follows that

$$\text{remaining } \angle ACB = \text{remaining } \angle DFE.$$

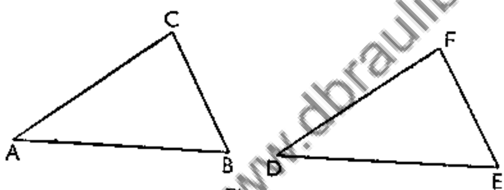


Fig. 8

Place $\triangle ABC$ on $\triangle DEF$ so that A falls on D and AB lies along DE.

Then B falls on E because $AB = DE$,

and AC lies along DF because $\angle BAC = \angle EDF$,

and BC lies along EF because $\angle ABC = \angle DEF$.

Since AC lies along DF, pt. C must fall somewhere on DF or DF produced, and since BC lies along EF, pt. C must fall somewhere along EF or EF produced. But as F is the only point common to both DF and EF, C must fall on F.

Thus the triangles completely coincide and are therefore equal in all respects.

Q.E.D.

This theorem may also be stated as follows. *If two triangles are equiangular and if a side of one is equal to the corresponding side of the other, the triangles are equal in all respects.*

Exercise VII (c)

Construct the following triangles:

1. Side AB = 2 in., ∠ABC = 60°, ∠BAC = 40°.
2. Side AB = 2 in., ∠ACB = 60°, ∠BAC = 40°.

Measure and compare the remaining elements. You will realize why for complete equality the triangles must have their **corresponding** sides equal.

Condition (*c*).—**If the three sides of one triangle are equal to the three sides of another triangle, each to each, the triangles are equal in all respects.**

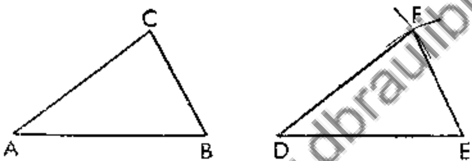


Fig. 9

Draw any △ABC and from it construct another △DEF having DE = AB, EF = BC, and DF = AC, starting by drawing a line DE equal to AB and using compasses to find the point F (fig. 9).

Given: △ABC and △DEF with AB = DE, AC = DF, BC = EF.

Prove: △ABC = △DEF in all respects.

Proof: Place △ABC on △DEF so that A is on D and AB lies along DE.

Then B falls on E because AB = DE Next, C lies on the arc of which D is the centre and DF (= AC) is the radius; also, C lies on the arc of which E is the centre and EF (= BC) is the radius. But F is the only point common to both arcs on that side of DE, and therefore C falls on F, and since the sides are straight lines, AC lies along DF and BC along EF. Thus the triangles coincide, and are therefore equal in all respects.

Hence ∠ABC = ∠DEF

∠BCA = ∠EFD

∠BAC = ∠EDF

and △ABC = △DEF in area. Q.E.D.

Exercise VII (d)

1. Construct the triangle whose sides are 3″, 4″, 5″. Measure the angles by protractor and compare them.
2. Construct the triangle whose sides are 2″, 3″, 3″. Measure and compare the angles.
3. Prove that the triangles formed by joining a pair of opposite corners of an oblong are equal in all respects.

PARTICULAR TRIANGLES

The Isosceles Triangle.

This triangle has two sides of equal length ("isosceles" means "equal-legged").

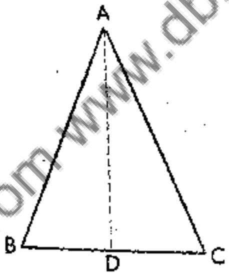


Fig. 10

In fig. 10, $\triangle ABC$ has $AC = AB$. Copy it on transparent paper, and fold the paper so that C falls on B and AC lies along AB, the crease (AD) running through A and cutting BC at D.

Observe that:

(i) The angle at C fits the angle at B, i.e. the angles opposite the equal sides are equal.

(ii) The crease (AD) bisects the angle at A.

(iii) The crease bisects the base BC and is at right angles to it.

PROPOSITION V

The angles opposite the equal sides of an isosceles triangle are equal.*

Given: $\triangle ABC$ with $AC = AB$ (fig. 10).

Prove: $\angle ABC = \angle ACB$.

Proof: Let $\angle BAC$ be bisected by straight line AD. (This does not affect the other angles.)

Then $\triangle ABD$ and $\triangle ACD$ have

$$AB = AC$$
$$AD = AD$$

and included $\angle BAD$ = included $\angle CAD$.

$\therefore$ $\triangle ABD = \triangle ACD$ in all respects, (CONDITION *a*)

and $\therefore$ $\angle ABD = \angle ACD$.

But $\angle ABD$ is the same as $\angle ABC$ and $\angle ACD$ as $\angle ACB$.

$\therefore$ $\angle ABC = \angle ACB$. Q.E.D.

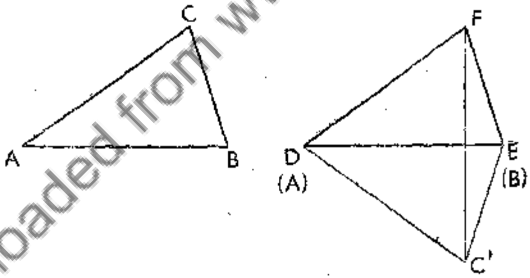


Fig. 11

Another proof for Condition (c) (p. 73).

Given: $\triangle ABC$ and $\triangle DEF$ with $AB = DE$, $AC = DF$, and $BC=EF$ (fig. 11).

Prove: $\triangle ABC = \triangle DEF$ in all respects.

* The proof of this proposition as given by Euclid (330–275 B.C.) was named "pons asinorum"—the bridge of asses—a bridge over which the dull-minded could not pass.

Proof: If one of the angles of $\triangle ABC$ is obtuse or right, let it be C. Place $\triangle ABC$ so that A falls on D, and AB lies along DE, and so that C falls on the side of DE opposite to F, i.e. at C′. Join C′F.

Pt. B falls on E because AB = DE.

Now since DF = DC′(AC), $\triangle DC'F$ is isosceles.

$$\therefore \angle DC'F = \angle DFC',$$

and since EF = EC′(BC), $\triangle EC'F$ is isosceles.

$$\therefore \angle EC'F = \angle EFC'.$$

By addition,

$$\text{whole } \angle DC'E = \text{whole } \angle DFE,$$
$$\text{i.e. } \angle ACB = \angle DFE.$$

In $\triangle ABC$ and $\triangle DEF$

since AC = DF and BC = EF

and incld. $\angle ACB$ = incld. $\angle DFE$,

$\triangle ABC = \triangle DEF$ in all respects by CONDITION (*a*). Q.E.D.

EXERCISE VII (E)

1. Write down the other equal elements of $\triangle ABD$ and $\triangle ACD$ (fig. 10).
2. On each side of a straight line BC construct an isosceles triangle. Name the other points A and D and join them by a straight line. Test whether AD bisects BC and the angles BAC and BDC, and is perpendicular to BC.
3. What kind of triangle is formed by joining the ends of two radii of a circle by a straight line?
4. On a base line 1″ long construct an isosceles triangle the equal sides of which are $1\frac{1}{2}$″ long.
5. Produce the equal sides of an isosceles triangle beyond the third side. Prove that the exterior angles formed are equal.
6. Produce one of the equal sides of an isosceles triangle beyond their common point. Prove that the exterior angle formed is double each of the equal angles of the triangle.
7. If a triangle ABC has $\angle ACB = \angle ABC$, what else would you expect to be equal?

8. On a base $1\frac{1}{2}''$ construct a triangle each of the other sides of which is also $1\frac{1}{2}''$.

The Equilateral Triangle.

An equilateral triangle has all its sides equal in length. It follows that all its angles also are equal since any two are equal. Each angle is therefore $\frac{180°}{3} = 60°$.

Make such a triangle and test the angles.

Exercise VII (f)

1. Starting with the angle of an equilateral triangle, construct angles of the following sizes, 30°, 15°, 45°, 90°, 75°.

   Test the results by protractor.

2. Divide a circle and its circumference into six equal parts by radii each making 60° with the next.

   Join the ends of these radii in order by chords. To what length is each chord equal?

   Which is the longer, an arc or the chord joining its ends?

   Which is the greater, the whole circumference or six radii?

   Which is the greater, the circumference or three times the diameter?

The Right-angled Triangle.

As the name implies, this triangle has one of its angles a right angle. The other two angles together equal a right angle. A very important relation exists between the sides of this triangle, the discoverer being the Greek philosopher Pythagoras (569–500 B.C.).

PROPOSITION VI

THEOREM OF PYTHAGORAS

In a right-angled triangle the sum of the squares on the sides forming the right angle is equal to the square on the side opposite the right angle. (The side opposite the right angle is called the *hypotenuse*.)

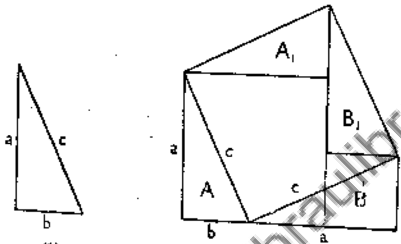
The truth of the theorem may be demonstrated as follows.

If the lengths of the sides of the right-angled triangle in fig. 12 (1) are a, b and c, c being the length of the hypotenuse, then it is required to show that

$$a^2 + b^2 = c^2.$$

In fig. 12 (2) the squares a^2 and b^2 are placed side by side, making an area equal to their sum.

The triangles marked A and B are each equal to the given triangle in fig. 12 (1).



(1) Fig. 12 (2)

If triangle A is turned to position A_1 and triangle B to position B_1, then from the area of the two squares the square on the hypotenuse c is built up;

$$\text{i.e. } a^2 + b^2 = c^2.$$

It follows that,
$$a^2 = c^2 - b^2$$
and
$$b^2 = c^2 - a^2.$$

The reader is advised to mark out the two squares and triangles on thin cardboard and to cut out and actually move and place the triangles as indicated.

Exercise VII (g)

1. Construct a right-angled triangle with the sides forming the right angle 3″ and 4″ respectively. Measure the hypotenuse and verify that the square of it is equal to the square of 3 plus the square of 4, that is, to $9 + 16 = 25$.
2. A rectangle is 2″ long and $1\frac{1}{2}$″ broad. What is the length of the diagonals?
3. Construct a square whose area is twice that of a given square.

4. By the method suggested by fig. 13, find the sides of squares respectively, twice, thrice, four times, etc., the area of the square on a line of unit length. The sides of these squares represent $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, etc. Check your drawing at $\sqrt{4}$.

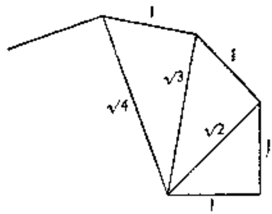


Fig. 13

5. Construct a square whose area is half that of a given square.
6. Prove that the bisector of any angle of an equilateral triangle divides the triangle into two equal right-angled triangles.
7. Find the remaining sides of a triangle whose angles are 30°, 60° and 90° when the shortest side is 10 cm. long.
8. Find the altitude of a square pyramid the side of the base of which is 4 cm. and the slant edge 6 cm.

PROPOSITION VII

The Area of a Triangle.

The area of a triangle is half that of the rectangle having the same base and altitude.

(Altitude is the height, as measured by a perpendicular to the base.)

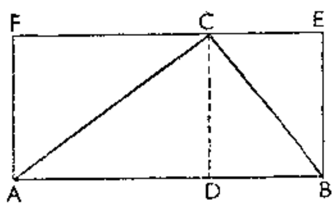


Fig. 14

ABC is a triangle, and ABEF the rectangle having the same base AB and altitude CD (fig. 14).

It is readily seen that △ADC is half of rectangle ADCF, ar △DBC is half of rectangle DBEC. Adding the triangles togeth and the rectangles together, the whole △ABC is half the who rectangle ABEF.

The area of a triangle is therefore half the product of the ba and the altitude.

It follows that triangles having equal bases and equal altitud are equal in area.

Note.—Any side may be regarded as the base, but the altitud must be measured by the perpendicular to that side (or that sid produced) from the opposite point of the triangle.

Exercise VII (h)

1. Show that the area of a right-angled triangle is half the produc of the sides forming the right angle.
2. What is the area of the right-angled triangle of Exercise VII (c) 1? Now regard the hypotenuse as the base, measure the altitude from the hypotenuse, and calculate the area from these data and compare the result with that calcu lated from the sides forming the right angle.

THE CIRCLE

Circumference and Diameter.—On p. 18, it is stated that on of the great problems of Mathematics is to find the relatio between the circumference and the diameter of a circle. It ha been long known that the number by which the diameter mus be multiplied to obtain the length of the circumference canno be determined exactly, but only approximately. Its decima part is unending.

Archimedes (287–212 b.c.), who might be called the very grea grandfather of Engineering, obtained a number often used to-da You can yourself get somewhere near the number by vario practical methods.

The following are a few:

1. Take a plate as nearly circular as you can get.

Measure the greatest distance across it at various places. F a perfectly circular plate these distances would be equal, of cour but there may be little differences, so find the average of thr or more measurements and take this average to be the diamet

Measure the circumference of the plate with a tape measure. Calculate, $\frac{\text{circumference,}}{\text{average diameter}}$, e.g. $\frac{31{\cdot}5''}{10''} = 3{\cdot}15$.

2. Get a wooden cylinder such as that used as a model for drawing.

Measure the diameter as before.

Place a band of paper round the cylinder so that it slightly overlaps. Make a pin-prick through the overlap. Remove the paper, flatten it out and measure the distance between the pin-pricks.

Calculate as in method 1.

3. Draw a circle of definite diameter, say $3\frac{1}{2}$ in. On a strip of transparent paper draw a long straight line. The object is to lay off the circumference of the circle as closely as possible on to this straight line.

Mark clearly one end-point of the straight line and a starting-point on the circumference (fig. 15). Place the transparent

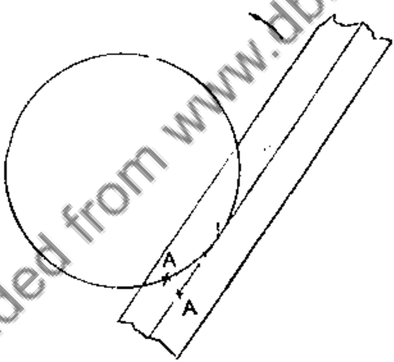


Fig. 15

paper so that these points are coincident, then using a pin or compasses point as pivot move the strip of paper so that a short piece of the straight line lies on a short arc of the circumference. Move the pivot to the other end of the short arc and in this way cover another short arc, continuing in short steps until the point of starting is again reached.

Mark the final point on the straight line clearly and measure the distance between the beginning and end points, and use it as the length of the circumference of the circle.

Calculate as before.

In these practical exercises you should obtain a number somewhere near 3·14. A more accurate number is 3·1416.

Archimedes found that the number lies between $3\frac{10}{71}$ and $3\frac{1}{7}$. You can use $3\frac{1}{7}$ without great error.

The Greek letter π (pi) is used to denote the number.

It should be memorized that:

(i) Circumference = diameter $\times 3\frac{1}{7}$.

(ii) Diameter = circumference $\div 3\frac{1}{7}$.

(iii) If c is the circumference and r the radius, then $c = 2\pi r$.

Area of a Circle.—It is evident from fig. 16 that the area of a circle is less than four times that of the square on a radius.

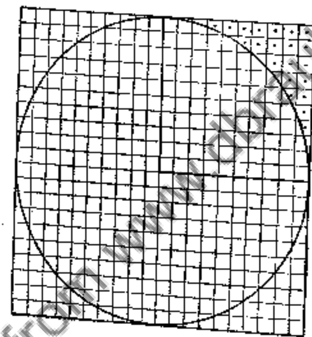


Fig. 16

The area cannot be calculated exactly, but if you reckon up the little squares in the circle (fig. 16) and divide the number by the number of little squares in the square on a radius you will get again somewhere near the number $3\frac{1}{7}$.

(The quickest way is to count the little squares in a corner piece, rejecting those less than half and counting as one any greater than half. Subtracting this number from the number of squares in the square on the radius, the number in a quadrant is obtained.)

Example: Number in a corner piece = 21.

" " a quadrant = 100 − 21 = 79.

" " the full circle = 79 × 4 = 316.

$$\frac{\text{Area of circle}}{\text{Area of square on a radius}} = \frac{316}{100} = 3{\cdot}16.$$

($21\frac{1}{2}$ in the corner piece gives 3·14, a better result.)

Remember then,

Area of circle $= 3\frac{1}{7}$ times the area of the square on a radius $= \pi r^2$.

EXAMPLE.—If the diameter of a circle is 6 in., the radius is 3 in. and the area $= 3\frac{1}{7} \times 3^2 = 3\frac{1}{7} \times 9 = 28\frac{2}{7}$ sq. in.

EXERCISE VII (I)

1. Find the circumference of each of the following circles:
   Diameter: (i) 10 cm., (ii) $10\frac{1}{2}$ in., (iii) 28 in.
   Radius: (iv) 1 ft., (v) 3·5 cm., (vi) $1\frac{3}{4}$ ft.
2. Find the area of the circles of Exercise 1.
3. The diameters of two circles differ by an inch. What is the difference in their circumferences?
4. The circumference of a wheel is 44". What is its diameter?
5. If the circumference of the earth at the equator is 25,000 miles, what is the diameter, assuming the equator to be circular?
6. Find the area of the circle of circumference 44 in.
7. Find the perimeter of a semicircle the straight side of which is 10 cm.
8. The perimeter of a semicircle is 36 in.; find its radius and area.
9. What is the area of a triangle whose base is equal in length to the circumference of a circle and whose altitude is equal to the radius (r) of that circle?
10. The perimeter of a figure is the distance round its boundaries. Draw the following figures, choose suitable symbols, a, b, c, etc., for the length of the sides, and write down in as simple a form as possible their perimeters.
    (i) A quadrilateral. (ii) A square. (iii) An oblong.
    (iv) A parallelogram. (v) A rhombus. (vi) A triangle.
    (vii) An isosceles triangle. (viii) An equilateral triangle.
11. Construct a semicircle and a quadrant of a circle, and write down the perimeter of each in terms of the radius.

12. Choose suitable symbols for the dimensions, and write down the area of each of the following:

(i) A rectangle. (ii) A square.
(iii) A parallelogram. (iv) A triangle.

13. A trapezoid has only one pair of opposite sides parallel. Construct one. Draw one of the diagonals, and the figure will be seen to consist of two triangles of the same altitude.

If the lengths of the parallel sides are denoted by a and b, and their distance apart by h, show that the area of the figure is $\frac{1}{2}h(a + b)$.

*14. Show that the area of a circle is $\frac{cr}{2}$, where c is the circumference and r the radius.

15. Obtain expressions for the area of the shaded portions of figs. 17, 18 and 19.

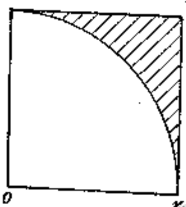


Fig. 17

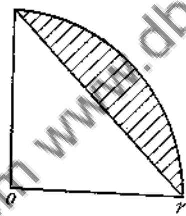


Fig. 18

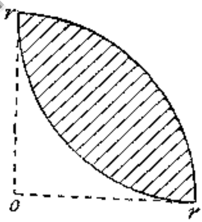
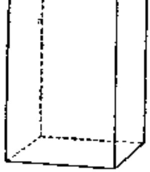
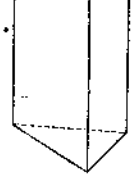


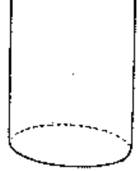
Fig. 19

16. The volume of a prism is the product of the area of the base and the altitude of the prism.

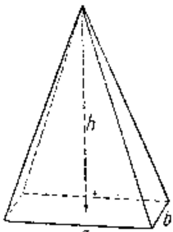
Choose suitable symbols for the dimensions, and state algebraically the volume of the following (figs. 20, 21, 22):


Rectangular Prism
Fig. 20

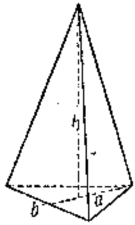

Triangular Prism
Fig. 21


Cylinder
Fig. 22

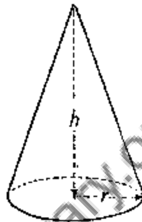
17. The volume of a pyramid is one-third that of the prism having the same base and altitude. Find the volume of the following (figs. 23, 24, 25):



Rectangular Pyramid
Fig. 23



Triangular Pyramid
Fig. 24



Cone
Fig. 25

CHAPTER VIII

SIMPLE EQUATIONS

1. Example i.—Consider the following little problem:

A bag of sugar (fig. 1) *and* 2 *pound weights in one pan of a balance, balance* 6 *pound weights in the other. What is the weight of the bag of sugar?* *

Now, in weighing, we usually put the things we are weighing alone in one pan.

What would be the effect of taking the 2 pound weights off the left-hand pan.

What would you do with the other side to restore the balance?

You would, of course, take 2 pound weights off.

The bag of sugar would then be seen to weigh 4 pounds.

This can be put down very conveniently in Algebra.

Write x for the number of pounds that the bag of sugar weighs.

It is now our business to find x.

We are told that $x + 2$ equals 6; written

$$x + 2 = 6.$$

* Packets of pen-nibs counterbalanced by loose pen-nibs will be found useful for practical exercises. The results can be verified by opening the packets.

This is called an equation, and from it we have to find the value of x.

To get x left alone on one side, subtract 2 from each side; then

$$x = 6 - 2;$$
$$\text{i.e.} \quad x = 4.$$

You will observe that if we consider the number 2 to have been carried to the other side of the equation, its sign has been changed from + to −. If we imagine it put back, the sign must be changed back from − to +. This is an important rule, viz.: *When an added or subtracted number is taken from one side of an equation to the other, its sign must be changed.*

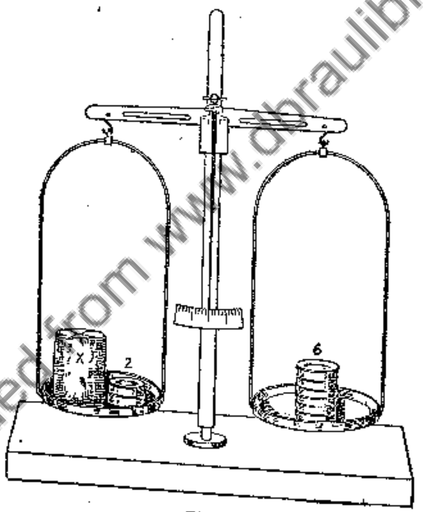


Fig. 1

2. Example ii.—*If* $x - 2 = 6$, *find* x.

The minus 2 can be got rid of by adding plus 2 to each side, thus:

$$x - 2 + 2 = 6 + 2,$$
$$\text{i.e.} \quad x = 6 + 2,$$
$$\text{i.e.} \quad x = 8.$$

Putting 8 instead of x in the original equation, the result is seen to be correct.

Here again we might have taken the -2 to the other side and changed it to $+2$.

3. Example iii.—Being told that three times a chosen number was 6, we should at once conclude that the chosen number was 2.

To state this in algebraic form, let the chosen number be represented by the symbol x; then

$$3x = 6.$$

The value of x is obtained at once by taking a third of 6,

$$\text{i.e.} \quad x = \tfrac{6}{3}$$
$$= 2.$$

Actually, both sides of the equation have been divided by 3, for $\frac{3x}{3} = x$. It is common experience that thirds of equal things are equal.

The result can be verified by substituting the value obtained in the original equation, thus:

When x is 2, $3x$ does equal 6.

Similarly, if $$12x = -60,$$
$$x = \frac{-60}{12},$$
$$\text{i.e.} \quad x = -5.$$

Check, $$12x = 12 \times -5 = -60.$$

Also, if $$-12x = -66,$$
$$x = \frac{-66}{-12},$$
$$\text{i.e.} \quad x = 5\tfrac{1}{2}.$$

Check, $$-12x = -12 \times 5\tfrac{1}{2} = -66.$$

Observe that the coefficient of x becomes a divisor on the other side of the equation.

4. Example iv.—An equation may have the symbol for the unknown number on both sides.

In such cases arrange the equation so that all such symbols are on the same side. Remember the rule given on p. 86.

$$5x - 13 = 3x + 5,$$
$$5x - 3x = 5 + 13,$$
$$2x = 18,$$
$$x = 9.$$

Check,

Left side: $5x - 13 = 5 \times 9 - 13 = 45 - 13 = 32.$

Right side: $3x + \ \ 5 = 3 \times 9 + \ \ 5 = 27 + \ \ 5 = 32.$

The sides are equal when x is 9.

Exercise VIII (a)

Find the value of the symbol in the following equations, and explain each step.

1. $6 = x + 2.$	2. $x - 3 = 6.$	3. $x - 3 = -6.$
4. $x + 5 = 5.$	5. $x - 5 = -5.$	6. $x + 5 = -5.$
7. $4 + x = 9.$	8. $4 - x = 9.$	9. $9 = 4 - x.$
10. $6x = -18.$	11. $-6x = 18.$	12. $-6x = -18.$
13. $3x = 10.$	14. $-3x = 10.$	15. $3x = -10.$
16. $-3x = -10.$	17. $4x = \frac{1}{2}.$	18. $-4x = \frac{1}{2}.$
19. $-4x = -\frac{1}{2}.$	20. $4x = -\frac{1}{2}.$	21. $6x + 5 = -13.$
22. $6x + 18 = 0.$	23. $-5x - 3 = 12.$	24. $2(x - 3) = 4.$
25. $x - 3 = 4 - (x - 3).$	26. $-7(x - 3) = 14.$	27. $3x - 11 = 4 - 2x.$

5. Example v.—*If* $\frac{1}{2}x = 6$, *find* x.

It is common knowledge that to whatever number half x is equal, x is equal to twice that number.

In this case $x = 12$.

Actually, both sides of the given equation have been multiplied by 2.

Thus:
$$\tfrac{1}{2}x \times 2 = 6 \times 2,$$
i.e.
$$x = 12.$$

Generally, *the equality is not destroyed by multiplying or dividing both sides of an equation by the same number.*

The operation performed above is really the same as that of Example iii.

Thus:
$$\tfrac{1}{2}x = 6;$$
$$\therefore\ x = \frac{6}{\frac{1}{2}} = 6 \times \tfrac{2}{1} = 12.$$

Similarly, if
$$-\tfrac{3}{5}x = 18,$$
$$x = 18 \times -\tfrac{5}{3},$$
i.e.
$$x = -30.$$

Check,
$$-\tfrac{3}{5}x = -\tfrac{3}{5} \times -30 = 18.$$

Notice that the 5, which is a divisor on the left, becomes a multiplier on the right, and that the 3, which is a multiplier on the left, becomes a divisor on the right.

6. Example vi.—The symbol need not necessarily be on the *left* side, thus:

If $$18 = 3x,$$

then $$\frac{18}{3} = x,$$

i.e. $$6 = x.$$

7. Example vii.

$$\frac{15}{x} = -3,$$

Multiply each side by x, or make x a multiplier on the right.

$$15 = -3x,$$

$$\frac{15}{-3} = x,$$

i.e. $$-5 = x.$$

Another method is to invert both sides of the equation, then:

$$\frac{x}{15} = -\frac{1}{3},$$

$$x = -\frac{15}{3},$$

i.e. $$x = -5.$$

Check, $$\frac{15}{x} = \frac{15}{-5} = -3.$$

8. Example viii.—*Solve the equation:*

$$3(x - 2) + 5 = \frac{3x - 5}{2} + 6.$$

Multiply both sides by 2, to get rid of the denominator 2.

$$6(x - 2) + 10 = (3x - 5) + 12,$$

Remove the brackets.

$$6x - 12 + 10 = 3x - 5 + 12,$$

Arrange the sides.

$$6x - 3x = 12 - 10 - 5 + 12,$$

$$3x = 9,$$

$$x = 3.$$

Check,

Left side: $3(x - 2) + 5 = 3(3 - 2) + 5 = 3 + 5 = 8.$

Right side: $\frac{3x - 5}{2} + 6 = \frac{3 \times 3 - 5}{2} + 6 = 2 + 6 = 8.$

Exercise VIII (b)

Find the value of the symbol in the following equations. Check each value.

1. $\frac{1}{3}x = 2$. 2. $\frac{2}{3}x = 2$. 3. $-\frac{2}{3}x = 2$. 4. $-\frac{2}{3}x = -2$.

5. $\frac{1}{3}x = \frac{1}{6}$. 6. $1\frac{1}{2}x = 6$. 7. $-1\frac{1}{2}x = 6$. 8. $-3{\cdot}5x = 14$.

9. $\frac{6}{x} = \frac{1}{2}$. 10. $\frac{6}{x} = 2$. 11. $-\frac{6}{x} = 2$. 12. $\frac{2}{x} = 6$.

13. $\frac{5}{x} + 3 = \frac{2}{x}$. 14. $\frac{6}{2x} = 2$. 15. $\frac{5}{3x} = \frac{5}{3}$.

16. $\frac{5}{3x} = \frac{3}{5}$. 17. $\frac{5x}{3} = \frac{3}{5}$. 18. $-\frac{7}{x} + 1 = \frac{2}{x} + 4$.

19. $1 + \frac{1}{2}x = 6$. 20. $\frac{1}{2}\left(4 - \frac{2}{x}\right) = 3$.

21. $2x - 78 = 23 - 3x$. 22. $3(x - 14) = 7(x - 18)$.

23. $7(5 - x) = 8(x - 5)$. 24. $\frac{2y + 12}{6} = \frac{7y + 108}{24}$.

25. $\frac{x + 36}{3} = \frac{2x}{5} + 3$. 26. $x - \frac{19 - 2x}{4} = \frac{2x + 11}{2}$.

27. $\frac{1}{3}\left(\frac{x}{8} - 2\right) - \frac{2}{3}\left(\frac{x}{6} - 4\right) = \frac{2}{9}\left(\frac{x}{4} - 6\right)$. 28. $\frac{3x}{2} - \frac{2x - 3}{3} = \frac{x - 3}{4}$.

29. $\frac{x}{3} - \frac{x}{4} + \frac{1}{6} = \frac{x}{8} + \frac{1}{12}$. 30. $\frac{1}{3x - 5} = \frac{2}{x + 5}$.

31. The following is a useful application of simple equations:

1000 farthings = 960 farthings + 40 farthings,
i.e. 1000 farthings = £1 + 10d.
Hence, 1 farthing = £·001 + ·01d. (dividing by 1000).

The cost of 100 articles at, say, $4\frac{1}{4}$d. each is readily calculated as follows:

$4\frac{1}{4}$d. = 17 farthings = £·017 + ·17d.

Hence, the required cost = £1·7 + 17d. = £1. 15s. 5d.

Similarly, find the cost of 100 articles at 2s. $4\frac{1}{2}$d. each. (Remember that a florin is £·1.)

32. Knowing that £·1 = a florin, £·05 = 1 shilling, and £·025 = 6d., express in decimal form:

(i) £3. 3s. $6\frac{1}{4}$d., (iii) £3. 17s. $6\frac{1}{2}$d.,
(ii) £3. 7s. $6\frac{1}{2}$d., (iv) £3. 7s. $8\frac{1}{4}$d.,

and quickly find the cost of 200 articles at these prices each.

33. The time (T) at any longitude (L) may be calculated from the time at Greenwich by means of the simple equation:

$T = G \pm \frac{L}{15}$, where T is the time required (in hours),
G is the time at Greenwich (in hours),
L is the longitude (in degrees).
When East, take the + sign.
When West, take the − sign.

Find the time at

(i) 45° E. when it is 8 a.m. at Greenwich.
(ii) 45° W. when it is 8 a.m. at Greenwich.
(iii) 75° W. when it is 12 (noon) at Greenwich.
(iv) 75° E. when it is 12 (noon) at Greenwich.
(v) 80° E. when it is 6 p.m. at Greenwich.

CHAPTER IX

MULTIPLICATION AND DIVISION OF EXPRESSIONS, SQUARE AND SQUARE ROOT

1. Multiplication.

It should be clearly realized that $(a + b)(c + d)$ means that each term of $(c + d)$ has to be multiplied by a and also by b and the products added.

Example i. $(a + b)(c + d) = a(c + d) + b(c + d)$
$= ac + ad + bc + bd.$

The operation might have been set out as follows:

$$\begin{array}{rl} & c + d \\ & a + b \\ \hline \text{Multiply by } a, & ac + ad \\ \text{Multiply by } b, & \qquad + bc + bd \\ \hline \text{Add,} & ac + ad + bc + bd \\ \hline \end{array}$$

For complex expressions the latter arrangement is preferable, but for simple expressions we shall adopt the former.

Example ii. $(a - b)(c - d) = a(c - d) - b(c - d)$
$= ac - ad - bc + bd.$

Example iii. $(x^2 - 3xy + 4y^2)(x^2 - 2xy - 2y^2)$.

$$\begin{array}{rl} & x^2 - 3xy + 4y^2 \\ & x^2 - 2xy - 2y^2 \\ \hline \text{Multiply by } x^2, & x^4 - 3x^3y + 4x^2y^2 \\ \text{Multiply by } -2xy, & \qquad - 2x^3y + 6x^2y^2 - 8xy^3 \\ \text{Multiply by } -2y^2, & \qquad\qquad - 2x^2y^2 + 6xy^3 - 8y^4 \\ \hline \text{Add,} & x^4 - 5x^3y + 8x^2y^2 - 2xy^3 - 8y^4 \\ \hline \end{array}$$

Notice that like terms are placed in the same column.

Exercise IX (a)

Multiply out the following products, and check your results by giving numerical values to the symbols:

1. $(a + b)(c - d)$.
2. $(a - b)(c + d)$.
3. $(2a + 3b)(3c + 2d)$.
4. $(2a + 3b)(3c - 2d)$.
5. $(2a - 3b)(3c - 2d)$.
6. $(3m^2 - 5n^2)(2p^3 - 3q^3)$.
7. $(x - 2)(x - 3)$.
8. $(x + 2)(x + 3)$.
9. $(x + 2)(x - 3)$.
10. $(x - 2)(x + 3)$.
11. $(2x + y)(2x - 3y)$.
12. $2(3x - 2)(2x - 3)$.
13. $a(x - a)$.
14. $x(x - a)(x + a)$.
15. $(a + b)(c + d + e)$.
16. $(a + b)(a - b + c)$.
17. $(a + 3)(b - 4)$.
18. $(2xy - 5)(5xy + 3)$.
19. $(x^2 + 3x - 1)(x - 2)$.
20. $(2a^2 - 3a + 2)(a^2 - 2a + 1)$.

21. $(x^2 + 3xy - 4y^2)(x^2 - 2xy - 3y^2)$.

22. $\left(x^2 + 1 + \frac{1}{x^2}\right)\left(x - \frac{1}{x}\right)$.

23. $(a^2 + b^2 + c^2 + bc - ac + ab)(a - b + c)$.

Test your answer by putting b and c each equal to a.

24. Multiply $2 + 3x - 4x^2 + 2x^3$ by $2 - x + x^2 - 3x^3$.

The following products, (1), (2) and (3), are very important, and must be memorized.

$$(1) \quad (x + y)(x + y) = x(x + y) + y(x + y)$$
$$= x^2 + xy + xy + y^2$$
$$= x^2 + 2xy + y^2.$$

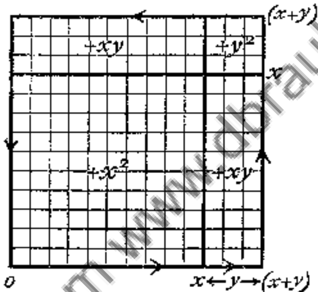


Fig. 1

This is, of course, the square of $(x + y)$, and therefore we can write:

$$(x + y)^2 = x^2 + 2xy + y^2.$$

Fig. 1 illustrates the result.

$$(2) \quad (x - y)(x - y) = x(x - y) - y(x - y)$$
$$= x^2 - xy - xy + y^2$$
$$= x^2 - 2xy + y^2,$$
$$\text{i.e. } (x - y)^2 = x^2 - 2xy + y^2$$

Notice that the expression obtained by squaring an expression of two terms (called a *binomial*) contains:

The square of the first term.
Twice the product of the two terms.
The square of the second term.

(3)
$$(x + y)(x - y) = x(x - y) + y(x - y)$$
$$= x^2 - xy + xy - y^2$$
$$= x^2 - y^2.$$

This result may be stated as follows:

The product of the sum of and the difference between two terms is equal to the difference between the squares of the terms.

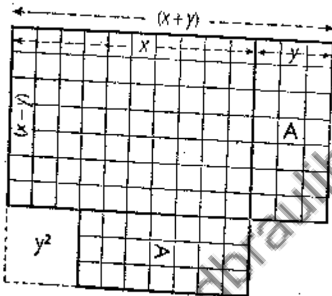


Fig. 2

Fig. 2 illustrates the relation. With oblong A in position A the area of the figure $= (x + y)(x - y)$, and in position ⊳, the area $= x^2 - y^2$.

Identity.

An equation such as:

$$(x + y)^2 = x^2 + 2xy + y^2,$$

which merely expresses the same quantity in two different forms, is called an *Identity*. It is true for all values of the symbols.

Contrast this with an equation like $x + 3 = 5$, which is true for particular values of x only—in this case 2.

Exercise IX (b)

Find:

1. $(a + b)^2$, $(a - b)^2$, $(a + b)(a - b)$.
2. $(2a + b)^2$, $(2a - b)^2$, $(2a + b)(2a - b)$.
3. $(a + 2b)^2$, $(a - 2b)^2$, $(a + 2b)(a - 2b)$.
4. $(2x + 5)^2$, $(2x - 5)^2$, $(2x + 5)(2x - 5)$.
5. $\left(\frac{x}{2} + 1\right)^2$, $\left(\frac{x}{2} - 1\right)^2$, $\left(\frac{x}{2} + 1\right)\left(\frac{x}{2} - 1\right)$.

6. $(2x + 3y)^2$, $(2x - 3y)^2$, $(2x + 3y)(2x - 3y)$.
7. $(10 + 7)^2$, $(50 - 3)^2$, $(50 + 5)(50 - 5)$, $(50 + \cdot 03)^2$.
8. $\{(a + b) + c\}^2$, $\{(a + b) - c\}^2$, $\{(a + b) + c\}\{(a + b) - c\}$.
9. $\{(2a - b) + 2c\}^2$, $\{(2a - b) - 2c\}^2$, $\{(2a - b) + 2c\}\{(2a - b) - 2c\}$.
10. $\{6(2a + b)\}^2$, $\{(a + b)(a - b)\}^2$.
11. Choose several different values for x and y, and verify that in every case

$$(x + y)^2 = x^2 + 2xy + y^2.$$

2. Division.

The arrangement for division of algebraic expressions is like that used in Arithmetic.

EXAMPLE.—*Divide* $6x^2 + 11xy - 10y^2$ *by* $2x + 5y$.

(i) For the first term of the answer, divide $6x^2$ by $2x$.

(ii) Multiply the whole of the divisor by $3x$, place the result under the dividend, and subtract.

(iii) For the next term, divide $-4xy$ by $2x$.

Continue as in (ii).

Repeat these operations until either there is no remainder or a remainder in which the first term is not divisible by the first term of the divisor.

$$\begin{array}{r|l} & 3x - 2y \quad (\textit{Quotient}) \\ 2x + 5y & 6x^2 + 11xy - 10y^2 \ (\textit{Dividend}) \\ & \underline{6x^2 + 15xy} \\ & -4xy - 10y^2 \\ & \underline{-4xy - 10y^2} \\ & \underline{\cdot \qquad \cdot} \end{array}$$

Notice that the dividend and divisor have their symbols in the same order.

The result may be checked either by multiplying the quotient and divisor, and comparing the product with the dividend, or by giving numerical values to the symbols, finding the values of the divisor, dividend and quotient, and checking by Arithmetic.

EXERCISE IX (c)

1. Verify the correctness of the worked example in Division.

Divide:

2. $a^2 + 2ab + b^2$ by $a + b$.
3. $a^2 - 2ab + b^2$ by $a - b$.
4. $a^2 - b^2$ by $a + b$.
5. $a^2 - b^2$ by $a - b$.

6. $a^2 + b^2$ by $a + b$.

7. $a^2 + b^2$ by $a - b$.

8. $x^2 - 5x + 6$ by $x - 3$.

9. $4a^3 - 8a^2 + 8a$ by $4a$.

10. $-15x^3y^3 - 5x^2y^2 + 20xy$ by $-5xy$.

11. $2x^3 - x^2 + 3x - 9$ by $2x - 3$.

12. $a^4 - 16b^4$ by $a + 2b$.

13. $12 + a - 5a^2 + a^3$ by $4 - a$.

14. $6x^4 - x^3y - x^2y^2 + 11xy^3 - 15y^4$ by $3x^2 - 2xy + 5y^2$.

15. $x^4 + 64$ by $x^2 - 4x + 8$.

16. $1 - a - 3a^2 - a^5$ by $1 - 3a + 2a^2 - a^3$.

17. $6x^4 - x^3 - 9x^2 + 9x - 5$ by $3x^2 - 2x + 1$.

What is the remainder? Now find for what value of x this remainder will equal 0.

18. Find c such that $x^2 + 5x + c$ is exactly divisible by $x + 2$.

3. The Square of a Binomial.

Given the first two terms of the square of a binomial to find the third term.

EXAMPLE. $x^2 - 6xy$.

(i) Since x^2 is the square of the first term of the binomial, the first term is x.

(ii) Since $-6xy$ is twice the product of the two terms, the product is $\frac{-6xy}{2}$, i.e. $-3xy$, and since one term is x, the other is $\frac{-3xy}{x}$, i.e. $-3y$.

(iii) The binomial is therefore $(x - 3y)$ and the complete square $x^2 - 6xy + 9y^2$.

More briefly: To find the third term, divide the second term by twice the square root of the first term and square the result. Thus:

$$\frac{-6xy}{2x} = -3y,$$

$$(-3y)^2 = 9y^2.$$

The process is illustrated in fig. 3.

The large rectangle represents $x^2 + ax$. Halve the rectangle which represents ax, by a straight line which bisects a. Place the upper half of the rectangle in the position indicated by the arrow. It is then seen that a square of side $\frac{1}{2}a$, and therefore of area $\frac{1}{4}a^2$, is required to complete the square, the side of which is $(x + \frac{1}{2}a)$.

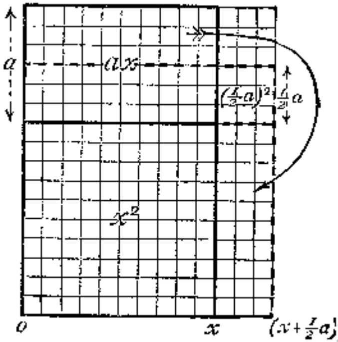


Fig. 3

Exercise IX (d)

Find the terms necessary to make each of the following expressions the square of a binomial. State the binomial in each case.

1. (i) $a^2 + 2ab$. (ii) $a^2 - 2ab$. (iii) $a^2 \ldots + b^2$.
2. (i) $x^2 + 4xy$. (ii) $x^2 - 4xy$. (iii) $x^2 \ldots + 4y^2$.
3. (i) $2ab + b^2$. (ii) $-2ab + b^2$. (iii) $-2b + b^2$.
4. (i) $4y^2 - 4xy$. (ii) $4x^2 + 4xy$. (iii) $4x^2 + 4x$.
5. (i) $9x^2 - 12xy$. (ii) $9x^2 + 12xy$. (iii) $9x^2 - 12x$.
6. (i) $x^2 + x$. (ii) $x^2 - x$. (iii) $x^2 - \cdot 8x$.
7. $49x^4 - 70x^2y^2$.
8. $a^2x^2 - 6ax$.
9. $a^2x^2 + 6axby$.
10. $9x^4 - 6x^2y$.
11. $16a^6 - 8a^3b$.
12. $\dfrac{9}{x^2} - \dfrac{12}{xy}$.

4. Square Root.

We have seen that:

$$(a + b)^2 = a^2 + 2ab + b^2.$$

It follows that the square root of $a^2 + 2ab + b^2$ is $a + b$,*

$$\text{i.e. } \sqrt{a^2 + 2ab + b^2} = a + b.$$

* We might take $-a - b$ as the square root, but it is usual to give only the root beginning with the positive sign.

Examining $a^2 + 2ab + b^2$, we see that:

(i) The first term (a) of the square root is the square root of the first term (a^2) of the expression.

(ii) The second term (b) of the square root is contained in the remaining part of the expression, $2ab + b^2$, which may be written $b(2a + b)$.

The process of finding square root is arranged as follows:

(i) The first term a, is the square root of a^2.

(ii) Subtract a^2 from the expression.

(iii) Form a new divisor by doubling what is in the answer (a), and adding the result (b) of dividing this double ($2a$) into the first term ($2ab$) of the line $2ab + b^2$.

(iv) Place the quotient (b) in the answer, multiply the new divisor by it, and complete the step as in division.

$$\begin{array}{r|l}
 & a + b \quad \textit{Answer.} \\
a & a^2 + 2ab + b^2 \\
 & a^2 \\
2a + b & \quad\; + 2ab + b^2 \\
 & \quad\; + 2ab + b^2 \\
 & \quad\; \cdot \qquad \cdot
\end{array}$$

General Example.—*Find* $\sqrt{9x^4 - 12x^3 - 2x^2 + 4x + 1}$.

First divisor,

$\sqrt{9x^4} = 3x^2$.

Second divisor,

$2 \times 3x^2 + \dfrac{-12x^3}{6x^2}$.

Third divisor,

$2(3x^2 - 2x) + \dfrac{-6x^2}{6x^2}$.

$$\begin{array}{r|l}
 & 3x^2 - 2x - 1 \quad \textit{Answer.} \\
3x^2 & 9x^4 - 12x^3 - 2x^2 + 4x + 1 \\
 & 9x^4 \\
6x^2 - 2x & \qquad - 12x^3 - 2x^2 + 4x + 1 \\
 & \qquad - 12x^3 + 4x^2 \\
6x^2 - 4x - 1 & \qquad\qquad\quad - 6x^2 + 4x + 1 \\
 & \qquad\qquad\quad - 6x^2 + 4x + 1 \\
 & \qquad\qquad\quad \cdot \qquad \cdot \qquad \cdot
\end{array}$$

The same method is used in Arithmetic.

Example i.—*Find* $\sqrt{116964}$.

$$\begin{array}{r|l}
 & \overset{\text{H.}}{3}\;\; \overset{\text{T.}}{4}\;\; \overset{\text{U.}}{2} \quad \textit{Answer.} \\
\overset{\text{H.}}{3} & 11\;69\;64 \\
 & \;9 \\
\overset{\text{H. T.}}{6\;4} & \;2\;69 \\
 & \;2\;56 \\
\overset{\text{H. T. U.}}{6\;8\;2} & \quad\;\; 13\;64 \\
 & \quad\;\; 13\;64 \\
 & \quad\;\;\; \cdot \;\; \cdot
\end{array}$$

Notice that:

(i) For every two digits in the given number there is one digit in the square root (but see remark (i) in Example (ii) below).

(ii) The divisor 64 is really twice 3 hundreds + 4 tens; i.e. 640, and the divisor 682, twice (3 hundreds + 4 tens) + 2 units.

EXAMPLE ii.—*Find* $\sqrt{678{\cdot}285}$.

(i) Mark the digits off in pairs from the decimal point. Notice that on the extreme left 6 stands alone, but that on the extreme right a nought is added to complete the pair.

(ii) Proceed as in Example i.

(iii) On bringing down 28 and trying 1 in the answer, 521 is obtained for the number to be subtracted. As this is greater than 228, place 0 in the answer *and in the divisor*, and bring down the next two digits.

```
             2  6 · 0  4   Answer.
        2 ) 6 78·28 50
            4
       46 ) 2 78
            2 76
     5204 )    2 28 50
               2 08 16
    10408 )      20 34
```

The answer 26·04 is correct to the second decimal place, for the third decimal figure will be found to be less than 5.

5. The Right-angled Triangle.

A knowledge of square root is necessary for the solution of problems referring to the sides of a right-angled triangle.

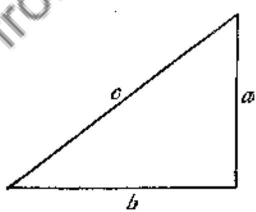


Fig. 4

If c is the hypotenuse of a right-angled triangle, and a and b the remaining sides (fig. 4), then

$$c^2 = a^2 + b^2,$$

from which

$$c = \sqrt{a^2 + b^2},$$

i.e. *the hypotenuse is equal to the square root of the sum of the squares of the remaining sides.* (Theorem of Pythagoras, p. 77.)

Example i.—*If $a = 3$ cm. and $b = 4$ cm., find c.*

$$c = \sqrt{a^2 + b^2} = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \text{ cm.}$$

Again, since $c^2 = a^2 + b^2$,

$$a^2 = c^2 - b^2 \quad\text{and}\quad b^2 = c^2 - a^2,$$

from which $a = \sqrt{c^2 - b^2}$ and $b = \sqrt{c^2 - a^2}$,

i.e. *a side, not the hypotenuse, is equal to the square root of the difference between the squares of the hypotenuse and of the remaining side.*

Example ii.—*If $c = 5$ cm. and $b = 4$ cm., find a.*

$$a = \sqrt{c^2 - b^2} = \sqrt{25 - 16} = \sqrt{9} = 3 \text{ cm.}$$

Example iii.—*If $c = 5$ in. and $a = 3$ in., find b.*

$$b = \sqrt{c^2 - a^2} = \sqrt{25 - 9} = \sqrt{16} = 4 \text{ in.}$$

Exercise IX (e)

Find the square roots of the following:

1. $4a^2 + 20ab + 25b^2$.
2. $4a^2 - 20ab + 25b^2$.
3. $16x^2 - 40xy + 25y^2$.
4. $36x^4 - 12x^2 + 1$.
5. $a^4 - 4a^3 + 8a + 4$.
6. $1 - 4y + 6y^2 - 4y^3 + y^4$.
7. $x^4 - 2x^3y + 5x^2y^2 - 4xy^3 + 4y^4$.
8. $9a^4 - 12a^3 + 10a^2 - 4a + 1$.
9. 9801.
10. 15129.
11. 3080·25.

Determine to the third decimal place:

12. $\sqrt{2}$.
13. $\sqrt{3}$.
14. $\sqrt{5}$.
15. $\sqrt{6}$.
16. $\sqrt{7}$.
17. $\sqrt{8{\cdot}263}$.
18. $\sqrt{0{\cdot}03856}$.
19. $\sqrt{231{\cdot}5}$.
20. $\sqrt{1583{\cdot}62}$.
21. Calculate the hypotenuses of the right-angled triangles of which the sides are:
    (i) 5 cm. and 12 cm.
    (ii) 7 cm. and 24 cm.
    (iii) 40 cm. and 9 cm.
    (iv) 13 in. and 84 in.
    (v) 33 ft. and 56 ft.
    (vi) 35 cm. and 12 cm.
    (vii) 2·4 in. and 3·6 in. (Answer to first decimal place.)

22. Find the remaining side of each of the following right-angled triangles.

(i)	Hypotenuse,	13 cm.;	one side,	5 cm.
(ii)	,,	73 cm.;	,,	48 cm.
(iii)	,,	117 cm.;	,,	45 cm.
(iv)	,,	29 in.;	,,	21 in.
(v)	,,	109 in.;	,,	91 in.
(vi)	,,	85 ft.;	,,	36 ft.
(vii)	,,	$3\sqrt{2}$ cm.;	,,	3 cm.
(viii)	,,	30·3 ft.;	,,	12·2 ft.

23. A ladder 40 ft. long is placed against a house so that its upper end just reaches a spout 35 ft. above the ground. How far is the foot of the ladder away from the wall?

24. Calculate to two places of decimals the length of the diagonal of a square of side 8 cm.

25. The adjacent sides of an oblong measure 10 cm. and 15 cm. Find the length of its diagonals.

26. Find, by the use of a right-angled triangle, the radius of a pipe which has a section equal to the sum of the sections of two given circular pipes.

CHAPTER X

GEOMETRY: ADDITIONAL FUNDAMENTAL THEOREMS

PROPOSITION IX

Angles of Rectilineal Figures.

If the figure has n sides the sum of its angles is (n − 2) straight angles.

Consider a figure like that of fig. 1. By drawing straight lines from one corner to each of the others, the figure is divided into triangles two less in number than the number of sides. The angles of these triangles make up the angles of the figure, and as the angles of each triangle together equal one straight angle, the total of the angles of all the triangles and therefore of the figure, is

$$(n - 2) \text{ straight angles}$$
$$\text{or, } (n - 2) \times 180 \text{ degrees.}$$

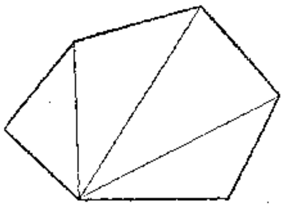


Fig. 1

Now in any rectilineal figure there are as many angles as sides. If all the angles are equal then the size of each is:

$$\frac{(n-2) \times 180}{n} \text{ degrees.}$$

Example.—The figure shown (fig. 1) has 6 sides, and therefore 6 angles.

The sum of its angles = 6 − 2 = 4 straight angles
= 720°.

If all the angles were equal, each would be $\frac{720°}{6} = 120°$.

Note.—Figures having all their sides equal and all their angles equal are called *regular figures*. Figures with many sides and therefore many angles are called *polygons*.

EXERCISE X (A)

1. Calculate the size of the angles of the regular rectilineal figures with sides from 4 to 10 in number.
Complete the table below.

REGULAR PLANE RECTILINEAL FIGURES

Name of figure	No. of		Sum of angles, in straight angles	Size of each angle, in degrees
	Sides	Angles		
Square	4	4	2	90°
Pentagon	5			
Hexagon	6	6	4	120°
Heptagon	7			
Octagon	8			
Nonagon	9			
Decagon	10			

2. If a regular figure had a million sides, to what angle would each of its angles be very nearly equal? To what known figure would such a figure approximate?

PROPOSITION X

If two angles of a triangle are equal, then the sides opposite these angles are equal.

This is the converse of Proposition V, p. 75.

(*Notice that what was given and what had to be proved, in the original proposition, are interchanged to form the converse.**)

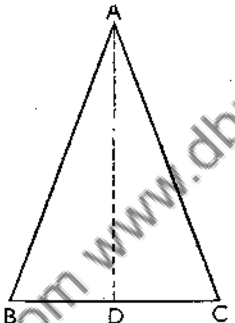


Fig. 2

Given: $\triangle ABC$ with $\angle ACB = \angle ABC$ (fig. 2).

Prove: $AB = AC$.

Proof: Let AD be the bisector of the remaining $\angle BAC$.

In $\triangle ABD$ and $\triangle ACD$

$$AD = AD$$
$$\angle BAD = \angle CAD$$
$$\angle ABD = \angle ACD.$$

$\therefore \triangle ABD = \triangle ACD$ in all respects . IV(*b*)

and $\therefore AB = AC$. Q.E.D.

* The student is warned that not all converses are true, though some are. All right angles are equal angles, but it is not true that equal angles are all right angles.

PROPOSITION XI

If one side of a triangle is greater than another, the angle opposite the greater side is greater than the angle opposite the less.

Given: $\triangle ABD$ with $AD > AB$.

Prove: $\angle ABD > \angle ADB$.

Proof: Consider first the isosceles $\triangle ABC$ (fig. 3) having $AC = AB$ and therefore $\angle ABC = \angle ACB$.

Imagine AC to increase in length by pt. C moving in the direction AC to, say, position D. Then $\angle ABC$ gets larger and since $\angle BAC$ is unchanged, and the sum of the three angles is always a straight angle, $\angle ACB$ must get smaller.

Therefore in triangle ABD, which has $AD > AB$, the angle opposite AD, namely, $\angle ABD >$ the angle opposite AB, namely, $\angle ADB$.

Q.E.D.

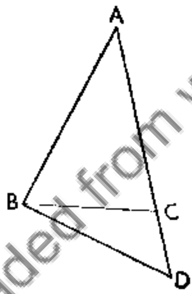


Fig. 3

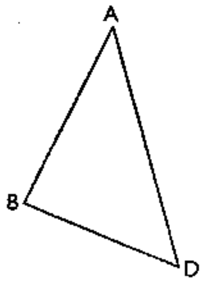


Fig. 4

PROPOSITION XII

If one angle of a triangle is greater than another, the side opposite the greater angle is greater than the side opposite the less.

This is the converse of the last proposition.

Given: $\triangle ABD$ with $\angle ABD > \angle ADB$ (fig. 4).

Prove: $AD > AB$.

Proof: The only possibilities are
(i) $AD = AB$; (ii) $AD < AB$; (iii) $AD > AB$.

(i) If AD = AB, then ∠ABD = ∠ADB.

But this is contrary to what is given.

∴ AD cannot equal AB.

(ii) If AD < AB, then ∠ABD < ∠ADB.

This also is contrary to what is given.

∴ AD cannot be less than AB.

Hence AD > AB, this being the only remaining possibility.

Q.E.D.

Note.—This form of proof is called "Method of Exhaustion".

Exercise X (b)

1. Referring to fig. 3, give reasons for each step of the following sequence and draw the final conclusion.
   ∠ABD > ∠ABC = ∠ACB > ∠CDB.
2. In the figure of Proposition XI, prove that
   ∠ABC = ∠CDB + ∠CBD.
3. Copy △ABD of the figure of Proposition XI on transparent paper and fold it so that AB lies along AD. It follows at once that ∠ABD > ∠ADB. Why?
4. Prove that in fig. 3 the difference between ∠ABD and ∠ADB is equal to twice ∠CBD.
5. Why is the hypotenuse the greatest side of a right-angled triangle?
6. Prove that the shortest line from a point to a given straight line is the straight line at right angles to it.

The Parallelogram.

Definition.—**A parallelogram is a four-sided figure with its opposite sides parallel.**

PROPOSITION XIII

A parallelogram has the following properties:

(i) **Its opposite sides are equal.**

(ii) **Its opposite angles are equal.**

(iii) **A diagonal divides it into two congruent triangles.**

(iv) **In area it is equal to the rectangle on the same base and having the same altitude.**

All these can be proved from the fact that the opposite sides are parallel.

Proof: In ▱ABCD (fig. 5) draw a diagonal, say DB.

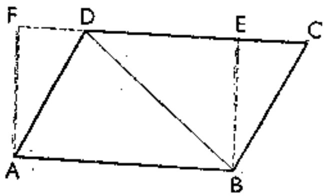


Fig. 5

Then △ADB and △DBC have

DB = DB

∠ADB = ∠DBC (alt. angles of parallels AD and BC)

∠ABD = ∠BDC. (alt. angles of parallels AB and DC)

∴ △ADB = △DBC in all respects,

i.e. AD = BC, AB = DC, ∠BAD = ∠DCB,

and the triangles are equal in area.

You can easily prove that ∠ADC = ∠ABC.

If ABEF is the rectangle of the same base and altitude, it is readily proved that △AFD = △BEC, and that rect. ABEF = ▱ABCD.

The area of ▱ABCD is therefore AB × BE,

i.e. base × altitude.

Exercise.—Prove that **the diagonals of a parallelogram bisect each other.**

PROPOSITION XIV

An angle at the centre of a circle is double any angle at the circumference standing on the same arc.

Given: In figs. 6, 7, ∠AOB has its vertex O at the centre of the circle and stands on the arc AB. ∠ACB stands on the same arc AB but has its vertex C on the circumference.

Prove: ∠AOB is double ∠ACB.

Proof: Join CO by a straight line and produce it to D.

Since AOC is an isosceles $\triangle$ with OA = OC,

$$\angle OAC = \angle OCA,$$

and since ext. $\angle AOD = \angle OAC + \angle OCA$ (opp. int. $\angle$s),

$$\angle AOD \text{ is double } \angle OCA.$$

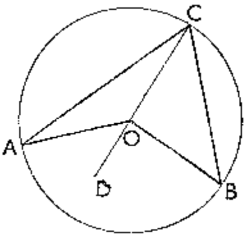


Fig. 6

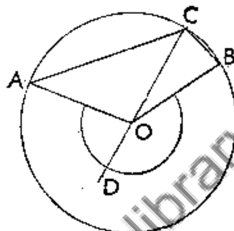


Fig. 6a

Similarly, ext. $\angle DOB$ is double $\angle OCB$.

By addition in fig. 6, and by subtraction in fig. 7,

$\angle AOB$ is double $\angle ACB$. Q.E.D.

Fig. 6*a* shows the case where the angle AOB is *reflex*, i.e. greater than a straight angle; the proof for fig. 6*a* is the same as for fig. 6.

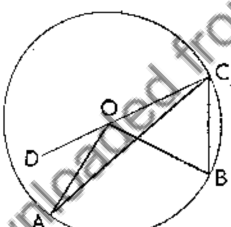


Fig. 7

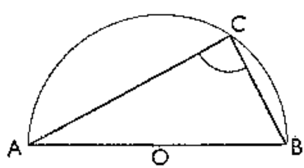


Fig. 8

PROPOSITION XV

As a special case of Proposition XIV:

Suppose AB is a diameter (fig. 8) and therefore $\angle AOB$ a straight angle, then $\angle ACB$ is a right angle. Therefore:

The angle in a semicircle is a right angle.

This useful fact should be remembered. Thales (640–546 B.C.) knew it.

Exercise X (c)

1. Applying Proposition XV, construct a right-angled triangle having its hypotenuse $2\frac{1}{2}$ in. long and one of the other sides $1\frac{1}{2}$ in. long.

   Show that the triangle may also be constructed by first of all laying down the side $1\frac{1}{2}$ in. long.

   Measure and write down the length of the remaining side.

2. In any right-angled triangle, prove that the line from the right angle to the mid-point of the hypotenuse is equal to half the hypotenuse.

PROPOSITION XVI

Angles in the same segment of a circle are equal to one another.

Referring to fig. 9, angles ACB, ADB and AEB are in the same segment, ACDEB, of the circle whose centre is O. They stand on the same arc AB.

They are equal because each is half the size of the angle at the centre, namely, ∠AOB, which stands on the same arc.

Q.E.D.

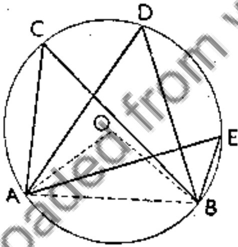


Fig. 9

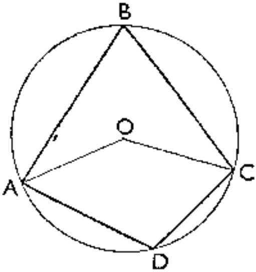


Fig. 10

PROPOSITION XVII

The opposite angles of a quadrilateral inscribed in a circle are together equal to a straight angle.

Given: Quadrilateral ABCD inscribed in a circle, centre O (fig. 10).

Prove: (i) ∠ABC + ∠ADC = a straight angle,
(ii) ∠BAD + ∠BCD = a straigh angle.

Proof: Join A and C to the centre O.

Then since

$$\angle ABC = \text{half of } \angle AOC$$

and $$\angle ADC = \text{half of reflex } \angle AOC,$$

$$\therefore \angle ABC + ADC = \text{half the sum of the two angles at O}$$
$$= \text{half of a cycle}$$
$$= \text{a straight angle.}$$

Since the four angles of the quadrilateral together equal two straight angles, it follows that the remaining pair, also, must equal a straight angle. Q.E.D.

PROPOSITION XVIII

The straight line joining the middle point of a chord of a circle to the centre is at right angles to the chord.

Given: Circle, centre O, and chord AB bisected at C (fig. 11).

Prove: Straight line OC is at right angles to AB.

Proof: Join OA and OB.

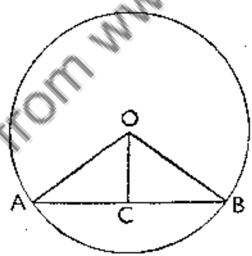


Fig. 11

Then, since △OAC and △OBC have

$$OA = OB$$
$$AC = CB$$
$$OC = OC,$$

$\therefore$ △OAC = △OBC in all respects. . . . IV(*c*)

$$\therefore \angle OCA = \angle OCB.$$

And since ACB is a straight angle, ∠OCA and ∠OCB are each a right angle. Q.E.D.

Exercise X (d)

1. Using the figure of Proposition XVIII prove by IV(b) that if OC is drawn at right angles to AB it bisects AB.
2. **To find the centre of a circle.**

   Draw two chords, not parallel. Bisect each by a straight line at right angles. The point of intersection of these bisectors is the centre. Test this yourself and use the method to find the centre of a circle to pass through the three points of a triangle.

PROPOSITION XIX

Tangent and Secant to a Circle.

A straight line just touching a circle (called a Tangent**) is at right angles to the radius drawn to the point of contact.**

A secant is a straight line cutting a circle. It cuts the circumference in two points.

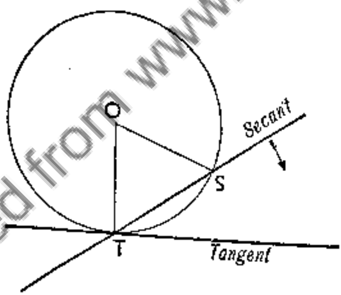


Fig. 12

Calling the points T and S (fig. 12), if they are joined to the centre O an isosceles $\triangle OST$ is formed which has

$$\angle OTS = \angle OST.$$

If $\angle TOS = x°$, then $\angle OTS = \frac{180 - x}{2} = \left(90 - \frac{x}{2}\right)°.$

Now imagine the secant to be pivoted at T and to be turned clockwise. The point S moves towards T, and $\angle TOS$ gets smaller and smaller. On the other hand, $\angle OTS$ gets larger and nearer and nearer to 90°. When S reaches and coincides with T, $\angle TOS$

closes, i.e. x becomes 0°, and the angle between the radius **OT** and the straight line through T, which now only touches the circle, becomes 90°, i.e. a right angle.

This fact concerning a tangent is important and should be remembered. It agrees with the statement in Exercise 6 (p. 105) that the shortest line to a given straight line from a point outside it, is the straight line at right angles to it.

Exercise X (e)

1. **To draw a tangent to a circle from (i) a point on the circumference; (ii) a point outside it.**

(i) This is easy, for it is only necessary to draw a radius to the point and then, at the point, a straight line at right angles to this radius.

(ii) To draw a tangent from a point outside the circle, it is easiest to make use of the fact that the angle in a semicircle is a right angle.

If O is the centre of the circle and P the point outside (fig. 13), join **PO** and on **PO** describe a semicircle. If T

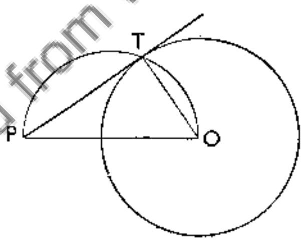


Fig. 13

is the point at which this semicircle cuts the given circle, the straight line joining PT is a tangent since ∠PTO is a right angle.

Questions.—How many tangents to the given circle can be drawn from P? How do their lengths compare?

2. To a circle of 3 cm. radius draw tangents from a point 5 cm. from the centre. Calculate their length.

3. How would you expect the tangents from a point to a sphere to compare in length?

4. Fig. 14 represents the internal sections of two pipes of given diameters.

Show that PQ represents the diameter of a pipe of which the area of section is the difference between those of

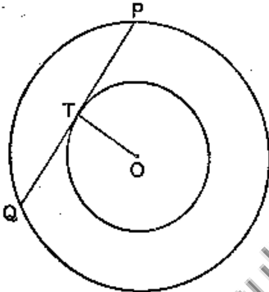


Fig. 14

the given pipes, and that the area of the ring between the two circumferences is $\pi(PT)^2$.

PROPOSITION XX

If the circumferences of two circles touch one another, internally or externally, the centres and the point of contact are in a straight line. (Two circles are said to touch one another if they have the same tangent at a point where they meet.)

If one circle is within the other, and P is the point of contact (fig. 15), then the tangent PT at P is a tangent to both circles.

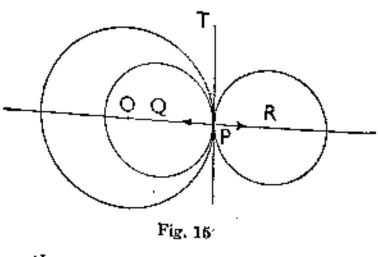


Fig. 15

If O and Q are the centres of the circles the radius OP and the radius QP are both at right angles to PT, i.e. $\angle TPO$ and $\angle TPQ$ are right angles, which is only possible if PO and PQ are in the same straight line.

Further, if R is the centre of a circle external to the others but touching them and the tangent at the same point P, then since ∠TPR also is a right angle, O, Q, P and R are in a straight line.

Q.E.D.

PROPOSITION XXI

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to any angle in the segment on the other side of the chord.

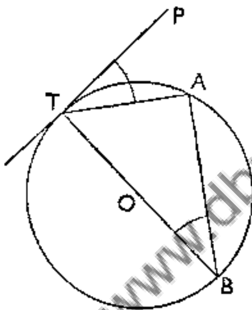


Fig. 16

Given: In fig. 16, tangent PT, and chord TA from the point of contact T. Draw the diameter TOB and join BA, then ∠TBA is an angle in the segment on the other side of TA.

Prove: ∠PTA = ∠TBA.

Proof: Since TAB is a semicircle, ∠TAB is a right angle.

∴ ∠TBA + ∠ATB = a right angle.

But since TP is a tangent and OT a radius,

∠PTB, i.e. ∠PTA + ∠ATB = a right angle.

∴ ∠PTA + ∠ATB = ∠TBA + ∠ATB,

and ∴ ∠PTA = ∠TBA.

By Proposition XVI,

∠TBA = any other angle in segment ABT.

Q.E.D.

Exercise.—Show that the angle between PT produced and TA is equal to any angle in the segment on the other side of TA.

CHAPTER XI

RATIO AND PROPORTION

1. Ratio.

A vulgar fraction, such as $\frac{3}{5}$, is sometimes called a ratio. The fraction $\frac{3}{5}$, when regarded as a ratio, is often written in the form 3 : 5 (read "3 to 5".) The first term, 3, is called the *antecedent* and the second, 5, the *consequent*.

A ratio represents the *relative* magnitude of quantities of the same kind.

If two lines, a and b, when measured by a scale of inches measure 3 in. and 5 in. respectively, then the ratio of the length of a to that of b is $\frac{3}{5}$.

This is a fixed relation between the lengths of these two lines, no matter by what scale we measure them. If they are measured by a scale which has half inches as units, then a will measure 6 units and b 10 units. The ratio of these lengths is $\frac{6}{10}$, and this is equal to the ratio $\frac{3}{5}$.

Ratios, like vulgar fractions, can be cancelled, or multiplied above and below, by the same number without altering their value.

When the antecedent is *greater* than the consequent, the ratio is said to be of *greater inequality*; when *less*, of *less inequality*.

When comparing ratios it will often be found convenient to work out the quotient of the terms in decimal form.

Exercise XI (a)

1. Express the following ratios in their simplest form:

$$\frac{10}{12},\quad \frac{81}{72},\quad \frac{10}{6},\quad \frac{1\frac{1}{2}}{2},\quad \frac{3\frac{1}{4}}{2\frac{1}{5}},\quad 15:25,\quad 8:12,\quad 3{\cdot}5:4{\cdot}2.$$

2. Draw any two straight lines, measure them in inches and also in centimetres, and write down the ratios of their lengths. Show as clearly as you can that the ratios are approximately equal.
3. What is the ratio of the perimeter of a square to one of its sides?
4. What is the ratio of the area of a square to that of the square of half the side?
5. What is the ratio of the value of a penny to the value of a shilling? Will your result represent the ratio of the weights also?

6. What is the ratio of the value of a florin to the value of a half-crown?

   If you have a balance and weights, see if the ratio of their weights is the same as that of their values.

7. What is the ratio of the area of a circle to that of the square on the radius, and also to that of the square on the diameter?

8. A circle is described to touch the sides of a square. What is the ratio of one of the corner areas to the whole square?

9. Arrange the following ratios in order of magnitude:

$$\tfrac{2}{5},\ \tfrac{1}{2},\ \tfrac{3}{7},\ \tfrac{5}{9},\ \tfrac{5}{8},\ \tfrac{6}{11}.$$

10. A bottle when filled with water weighs W gm., and when filled with milk M gm. If the empty bottle weighs B gm., what is the ratio of the weight of milk to the weight of an equal volume of water?

2. Proportion.

Proportion is the equality of ratios.

Draw two straight lines of length 3 in. and 6 in. respectively. The ratio of their lengths is $\frac{3}{6}$, or $\frac{1}{2}$. Draw two other lines of length 8 and 16 in. Their ratio is $\frac{8}{16}$, or again $\frac{1}{2}$. Their ratios are therefore equal. We can write

$$\frac{3}{6} = \frac{8}{16}.$$

Four numbers so related are called proportionals.

The first and last terms are called extremes, the second and third (6 and 8) means.

The fourth term (16) is called the fourth proportional of 3, 6 and 8.

A proportion is sometimes written in the form $3:6::8:16$ or more usually $3:6 = 8:16$ (read 3 is to 6 as 8 is to 16).

If any one of the four numbers is unknown, it can be readily calculated.

Suppose that we did not know the length of the first line, but that we knew the lines to be proportional.

Let x = the unknown length.

Then
$$\frac{x}{6} = \frac{8}{16},$$
from which
$$x = 3.$$

The four quantities need not be of the same kind. E.g. the value of gold is proportional to its weight. That is, the ratio of two weights is equal to the ratio of the corresponding values.

Thus, if 2 oz. are worth £15, then 6 oz. are worth £45.

The proportion is $$\frac{2}{6} = \frac{15}{45}.$$

Such a proportion is called a *direct* proportion because the ratios are directly equal. Contrast this with the following example of *inverse* proportion.

It is a truism that the faster a train travels, the less time it takes to cover a certain distance.

Suppose a train, A, goes at 40 miles an hour, and another, B, at 30 miles an hour. Then to travel a particular distance, say 240 miles, A takes 6 hr. and B takes 8 hr.

Now, $$\frac{\text{Speed of A}}{\text{Speed of B}} = \frac{40}{30},$$

and $$\frac{\text{Time of A}}{\text{Time of B}} = \frac{6}{8}.$$

A glance at the ratios shows that they are not directly equal, i.e. when both numerators refer to A and both denominators to B. But if the second ratio be inverted, we have two equal ratios and therefore a proportion.

$$\frac{\text{Speed of A}}{\text{Speed of B}} = \frac{40}{30} = \frac{8}{6} = \frac{\text{Time of B}}{\text{Time of A}},$$

i.e. the ratio of the speeds is equal to the inverse ratio of the times.

This is an *inverse* proportion.

3. Important Deductions in Proportion.

1. If $\frac{a}{b} = \frac{c}{d}$, then $$\frac{b}{a} = \frac{d}{c}.$$

Divide 1 by each ratio; then $$\frac{1}{\frac{a}{b}} = \frac{1}{\frac{c}{d}},$$

i.e. inverting and multiplying, $$\frac{b}{a} = \frac{d}{c}.$$

2. If $\frac{a}{b} = \frac{c}{d}$, then $$ad = bc.$$

Multiply both sides by bd (product of the denominators):

$$\frac{a}{b} \times bd = \frac{c}{d} \times bd, \quad \text{i.e. } ad = bc.$$

In words, the product of the extremes is equal to the product of the means.

Special case. If the means are equal, the product of the extremes is equal to the square of one of the means. E.g. if $\frac{a}{b} = \frac{b}{c}$, then $ac = b^2$.

c is called the *third proportional* to a and b, and b the *geometric mean* of a and c.

3. If $\frac{a}{b} = \frac{c}{d}$, then $\quad \frac{a}{c} = \frac{b}{d}.$

Since $\frac{a}{b} = \frac{c}{d}$, $\quad ad = bc.$

Divide both sides by cd; then $\frac{ad}{cd} = \frac{bc}{cd}$, $\quad \therefore \frac{a}{c} = \frac{b}{d}.$

4. If $\frac{a}{b} = \frac{c}{d}$, then $\quad \frac{a+b}{b} = \frac{c+d}{d}.$

Add 1 to each side; then $\frac{a}{b} + 1 = \frac{c}{d} + 1$,

from which $\quad \frac{a+b}{b} = \frac{c+d}{d}.$

5. If $\frac{a}{b} = \frac{c}{d}$, then $\quad \frac{a-b}{b} = \frac{c-d}{d}.$

Subtract 1 from each side; then $\frac{a}{b} - 1 = \frac{c}{d} - 1$,

from which $\quad \frac{a-b}{b} = \frac{c-d}{d}.$

6. If $\frac{a}{b} = \frac{c}{d}$, then $\quad \frac{a-b}{a+b} = \frac{c-d}{c+d}.$

Divide result 5 by result 4; b cancels on one side, and d on the other.

$$\therefore \frac{a-b}{a+b} = \frac{c-d}{c+d}.$$

7. If $\frac{a}{b} = \frac{c}{d}$, then $\quad \frac{a}{b} = \frac{a+c}{b+d} = \frac{a-c}{b-d}.$

Let $\frac{a}{b} = k$, then $\frac{c}{d} = k$,

and $$kb = a$$

and $$kd = c.$$

Adding $$k(b + d) = a + c,$$

$$\therefore k = \frac{a + c}{b + d};$$

$$\therefore \frac{a}{b} = \frac{a + c}{b + d}.$$

By subtracting instead of adding,

$$\frac{a}{b} = \frac{a - c}{b - d},$$

$$\therefore \frac{a}{b} = \frac{a + c}{b + d} = \frac{a - c}{b - d}.$$

Exercise XI (b)

Find x in the following proportions:

1. $\frac{x}{9} = \frac{25}{27}$. 2. $\frac{9}{x} = \frac{15}{4}$. 3. $\frac{5}{8} = \frac{x}{14}$. 4. $\frac{5}{8} = \frac{14}{x}$.

5. $\frac{x}{a} = \frac{b}{c}$. 6. $\frac{a}{x} = \frac{b}{c}$. 7. $\frac{a}{b} = \frac{x}{a}$. 8. $\frac{a}{b} = \frac{a}{x}$.

9. Examine the following ratios, and determine whether they are directly or inversely equal:

(i) $\frac{6}{10}$ and $\frac{9}{15}$. (ii) $\frac{8}{12}$ and $\frac{36}{24}$.

(iii) $\frac{12}{10}$ and $\frac{2 \cdot 4}{2}$. (iv) $\frac{10}{35}$ and $\frac{28}{8}$.

10. The following ratios are equal in pairs. Find the missing terms.

(i) $\frac{?}{5}$ and $\frac{6}{10}$. (ii) $\frac{?}{5}$ and $\frac{10}{6}$.

(iii) $\frac{4}{?}$ and $\frac{12}{36}$. (iv) $\frac{9}{2}$ and $\frac{?}{5}$.

11. Say whether the following are direct or inverse proportions:

(i) Circumference of a wheel A, is 10 ft.
Number of turns in a fixed distance = 528.
Circumference of a wheel B, is 12 ft.
Number of turns in the same distance = 440.

(ii) Circumference of a wheel A, is 10 ft.
Distance covered in a number of turns = 500 ft.
Circumference of a wheel B, is 12 ft.
Distance covered in the same number of turns = 600 ft.

12. An experiment showed that the weights of pieces of the same sheet of drawing paper were directly proportional to the areas of the surfaces. A portion having the shape of the map of Ireland weighed 3·052 gm., and another piece in the shape of a rectangle, the sides of which represented 235·2 and 292·8 miles on the same scale as the map, weighed 6·402 gm. Calculate the area of Ireland.

13. The radius of a circular arc is 5 cm. If the arc subtends an angle of 30° at the centre, calculate its length.

14. Compare the area of a sector, the angle of which is 60°, with the area of the whole circle of which it is a part.

15. A sector has a radius R and an angle $x°$. What fraction of the area of the whole circle is the area of this sector?

Find the formula for the area of the sector.

16. Take a point P within a circle and through P draw a number of chords, and a diameter of the circle.

Measure the segments of the chords and form a table, thus:

	Segments on left of diameter	Segments on right of diameter
1st Chord, 2nd " Etc.	0·4 in. 0·6 in.	1·8 in. 1·2 in.

Examine ratios of these numbers, and draw your conclusions.

17. In an experiment on the inclined plane, the following numbers were obtained:

Weight raised	Effort applied	Length of plane	Height of upper end of plane
278 gm.	68 gm.	52 cm.	13·5 cm.
278 gm.	90 gm.	52 cm.	17 cm.
278 gm.	105 gm.	52 cm.	20 cm.
278 gm.	140 gm.	52 cm.	26 cm.
278 gm.	161 gm.	52 cm.	30 cm.

Compare the ratios $\frac{\text{Effort}}{\text{Weight raised}}$ and $\frac{\text{Height}}{\text{Length}}$ of plane.

State your conclusion in algebraic form. What effort will be necessary to raise the weight when the height of the upper end of the plane is 24 cm.?

18. Find the arc between the ends of two radii of a circle, which make an angle of 150° and are 3 in. in length.

19. The arc of a sector of a circle of 5 in. radius measures 5 in.; calculate the angle between its bounding radii. This angle is double any angle standing on this arc, but with its vertex anywhere on the remaining part of the circumference of the circle. What is the angle in the latter case?

20. How many minute spaces does the large hand of a clock gain on the small hand in x min.?

21. A ball of copper weighs A gm. in air and W gm. when totally submerged in water. What is the difference in weight?

This difference is exactly equal to the weight of the liquid displaced by the ball. Express as a ratio, the weight of the ball compared with the weight of an equal volume of water.

22. A metal ball weighs A gm. in air, W gm. in water (totally submerged) and T gm. when totally submerged in turpentine.

Express as a ratio the weight of turpentine compared with the weight of an equal volume of water.

23. Write down as many new proportions as you can from:

$$\frac{3x}{2y} = \frac{5a}{8b}.$$

24. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{a} = \frac{c+d}{c}$, and that $\frac{a-b}{a} = \frac{c-d}{c}$.

25. The rims of two wheels revolve in contact without sliding on each other. If the diameter of one wheel is 12 in. and of the other 3 in., find how many times the smaller will turn when the larger makes one revolution. How do the revolutions depend upon the diameters?

26. Two wheels are geared together by means of a belt. If there is no slipping, show that the number of revolutions made by the wheels in the same time is inversely proportional to the diameters of the wheels.

27. A wheel having 12 teeth is geared to a wheel with 36 teeth. What will be the ratio of their revolutions in the same time?

4. Similar Triangles.

Draw a straight line AB, say 3 in. long, and from the end A draw another straight line AX, making an angle with AB (fig. 1).

Along AX, with a pair of compasses step off, say, 5 equal lengths.

Draw a straight line from the point marked 5 to the end B,

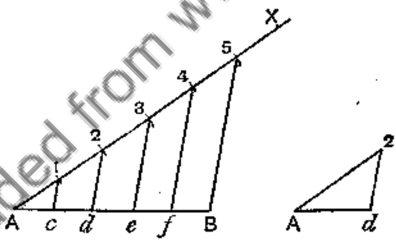


Fig. 1

and from the remaining marked points draw straight lines parallel to 5B to cut AB. Mark the points of intersection *c*, *d*, *e*, *f*.

Now measure the lengths A*c*, *cd*, *de*, *ef*, *f*B.

What is your conclusion?

Still using compasses, compare the lengths of *c*1, *d*2, *e*3, *f*4, B5.

Now look at the triangles AB5 and, say, A*d*2.

They have the same angle A; the angle at *d* is equal to the angle at B, and the angle at 2 is equal to the angle at 5. They have the same shape.

One is, however, bigger than the other, but there is a very special relation between their sides.

Find the following ratios:

$$\frac{\text{side (AB)}}{\text{side (A}d\text{)}}, \quad \frac{\text{side (A5)}}{\text{side (A2)}}, \quad \frac{\text{side (B5)}}{\text{side (}d\text{2)}}.$$

Your conclusion is, that all these ratios are equal.

Such triangles are said to be similar.

Notice that the terms of each ratio are opposite equal angles.

Sides opposite equal angles are called corresponding sides. In similar triangles the ratios of pairs of corresponding sides are equal, i.e.

$$\frac{AB}{Ad} = \frac{A5}{A2} = \frac{B5}{d2}.$$

The triangles need not, of course, be one inside the other, but may be quite apart, as shown in the figure.

The point to remember is, that they must be equiangular.

The ratios may be written so that the terms of a ratio refer to the same triangle.

Thus: since
$$\frac{AB}{Ad} = \frac{A5}{A2},$$
$$\therefore \frac{AB}{A5} = \frac{Ad}{A2}.$$

Observe that the sides forming the terms of these ratios contain equal angles.

5. Applications.

(I) The sides AB, BC and AC of a triangle ABC (fig. 2) are 3·5, 3·0 and 4·7 cm. respectively.

A line DE is drawn parallel to the base BC from a point D in AB, 2 cm. from A. Find the length of DE and of AE.

Let DE = x and AE = y.

Since triangles ADE and ABC are similar,

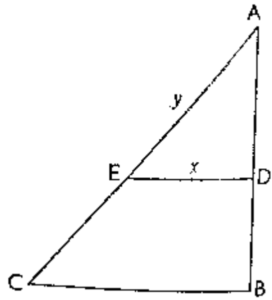


Fig. 2

$$\therefore \frac{x}{3} = \frac{2}{3{\cdot}5} \quad \text{and} \quad \frac{y}{4{\cdot}7} = \frac{2}{3{\cdot}5}.$$

From these simple equations x and y are readily calculated. Find them, and check your result by measurement.

(2) ABC is a right-angled triangle, the right angle being at C (fig. 3). **To prove** by proportion the relation between the sides, namely, that $a^2 + b^2 = c^2$.

Proof: Draw CD $\perp$ to AB.

Then since $\angle CAD = \angle BCD$,

and $\angle ACD = \angle CBD$,

and the other angles are right angles, therefore $\triangle ACD$, $\triangle CBD$ and $\triangle ABC$ are all similar.

If BD = x, then AD = $(c - x)$.

From $\triangle ABC$ and $\triangle CBD$

$$\frac{BC}{AB} = \frac{BD}{BC},$$

i.e. $\frac{a}{c} = \frac{x}{a}$, from which $a^2 = cx$. . . . . . . (i)

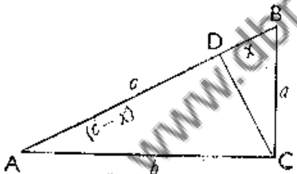


Fig. 3

From $\triangle ABC$ and $\triangle ACD$

$$\frac{AC}{AB} = \frac{AD}{AC},$$

i.e. $\frac{b}{c} = \frac{c - x}{b}$, from which $b^2 = c^2 - cx$. . . . (ii)

Adding relations (i) and (ii), $a^2 + b^2 = c^2$.

This is, of course, the theorem of Pythagoras.

(3) **AB and CD are chords of a circle intersecting at P. To show that PC×PD=PA×PB.**

Since the triangles APC and DPB (fig. 4*a*) are similar (angles in the same segment and therefore equal are indicated by the same number),

$$\therefore \frac{PA}{PC} = \frac{PD}{PB}$$

and

$$\therefore PC \times PD = PA \times PB,$$

i.e. the products of the parts of the chords are equal.

If AB is a diameter and CD a chord at right angles to it (fig. 4*b*), then since PC = PD

$$PC^2 = PA \times PB,$$

i.e. the square of half the chord is equal to the product of the parts of the diameter.

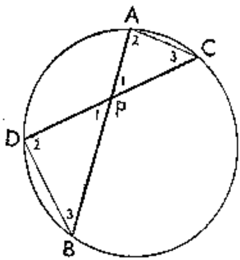


Fig. 4*a*

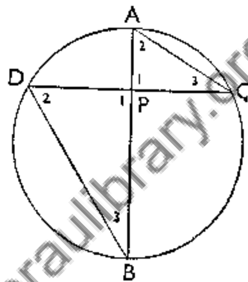


Fig. 4*b*

(4) **PT is a tangent to a circle from a point P (fig. 5) and PAB any secant from P. To show that $PT^2 = PA \times PB$.**

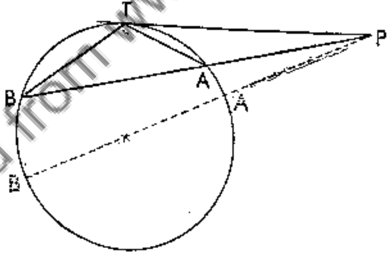


Fig. 5

The triangles PAT and PTB (fig. 5) are similar,

since $\angle TPA = \angle TPB$ . . . . same angle

and $\angle PTA = \angle PBT$ . . . . Prop. XXI

and $\therefore \angle PAT = \angle PTB$.

$$\therefore \frac{PA}{PT} = \frac{PT}{PB},$$

from which

$$PT^2 = PA \times PB,$$

i.e. the square of the tangent is equal to the product of the distances from P to the points at which the secant cuts the circumference of the circle.

An important case is when PAB passes through the centre (shown dotted); then AB is a diameter.

(5) **The areas of triangles of the same altitude are proportional to their bases.**

Let the altitude be a, and the bases respectively, b and c.

Denoting the areas of the triangles by A_1 and A_2, we have, since $A_1 = \frac{1}{2}ab$, and $A_2 = \frac{1}{2}ac$,

$$\frac{A_1}{A_2} = \frac{1}{2}ab \Big/ \frac{1}{2}ac = \frac{b}{c},$$

i.e. the ratio of the areas equals the ratio of the bases.

(6) **The bisector of an angle of a triangle divides the opposite side in the ratio of the other sides.**

Given: $\triangle ABC$ with $\angle BAC$ bisected by AD (fig. 6).

Prove: $\frac{BD}{DC} = \frac{AB}{AC}$.

Proof: Since $\triangle ADB$ and $\triangle ADC$ have the same altitude,

$$\frac{\triangle ADB}{\triangle ADC} = \frac{BD}{DC}.$$

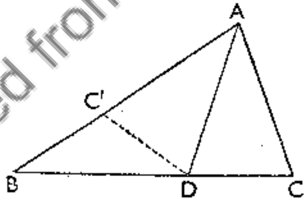


Fig. 6

Fold $\triangle ADC$ over AD, then C falls on AB, say at C′, since $\angle DAC = \angle DAB$. Now $\triangle ADB$ and $\triangle ADC'$ have the same altitude from D and their bases are AB and AC′.

$$\therefore \frac{\triangle ADB}{\triangle ADC'} = \frac{AB}{AC'} = \frac{AB}{AC}. \quad \text{Hence } \frac{BD}{DC} = \frac{AB}{AC}.$$

Q.E.D.

(7) **The medians of a triangle trisect each other.**

A median is the straight line drawn from an angle point of a triangle to the middle point of the opposite side. It divides the triangle into two triangles having the same altitude and equal bases and therefore equal in area.

Given: △ABC (fig. 7).

Prove: The medians trisect.

Proof: Let G be the point of intersection of any two medians AD and BE. Join GC.

Since AE = EC,

△ABE = △EBC (common vertex B)

and △AGE = △EGC. (common vertex G)

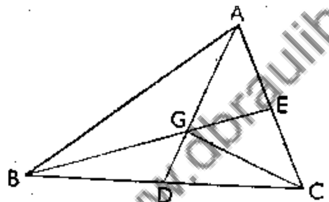


Fig. 7

By subtraction, △BGA = △BGC.

Again, since BD = DC,

△GDB = △GDC. (common vertex G)

∴ △BGC is double △GDB

and ∴ △BGA is double △GDB.

Since these △s have the same altitude from B, the base AG is double the base GD;

i.e. AD is trisected at G.

By subtracting △GDB from △ADB and △GDC from △ADC, it can be shown that BG is double GE.

∴ AD and BE trisect each other at G.

Similarly, it can be shown that the median from C trisects AD and BE at the same same point G.

Notes. Q.E.D.

You will notice that G is at one-third of the length of each median measured from the middle point of the opposite side.

The point G is the centre of area (sometimes called the centre of gravity) of △ABC.

Exercise XI (c)

1. In fig. 2 show that $\frac{AE}{EC} = \frac{AD}{DB}$.
2. Draw any triangle, and from any point on one side draw a straight line parallel to the base to intersect the remaining side. Letter the figure, and write down all the pairs of equal ratios you can find. Verify your statements by careful measurements.
3. Draw two straight lines x and y, 1·8 in. and 3 cm. long respectively. Taking the unit to be a straight line an inch long, find lines to represent

   (i) xy. (ii) $\frac{x}{y}$. (iii) $\frac{y}{x}$. (iv) x^2. (v) y^2.

   Measure the lines in inches, and check the results.
4. Using the lines in Ex. 3 for x and y, find a line representing $\frac{x + 3y}{7}$.
5. Taking an inch as the unit, find straight lines to represent

   (i) $2{\cdot}3 \times 0{\cdot}8$. (ii) $\frac{2{\cdot}3}{0{\cdot}8}$. (iii) $\frac{0{\cdot}8}{2{\cdot}3}$.

   Check by measurement.
6. When the shadow of a vertical stick 6 ft. long measures 8 ft., that of a building measures 75 ft. Find the height of the building.

   Thales (600 B.C.) used this method to find the height of the Pyramids of Egypt.
7. If, in fig. 4*b*, AB is 5 in. and AP 1 in., calculate PC.
8. If, in fig. 4*b*, PC is 6 cm. and AP 4 cm., find AB.
9. In fig. 5, let AB be a diameter. From T draw the chord at right angles to AB, and let it intersect AB at Q. Show that
$$PA \times QB = AQ \times PB.$$
10. The sides of a triangle are, respectively, 10, 14, and 18 cm. Find how the bisector of each of the angles divides the opposite side.
11. Verify by accurate construction of fig. 7 (p. 126) that CG produced bisects the side AB.
12. A ball, diameter 6 cm., rests inside a conical wine-glass of depth 8 cm. and rim diameter 12 cm. How far is the lowest point of the ball above the bottom of the glass?

CHAPTER XII

SPECIAL RATIOS, TRIGONOMETRY

1. The ratios of the sides of a right-angled triangle are of special importance in a branch of Mathematics, called Trigonometry.*

Draw a right-angled triangle ABC, having the angle A say 40° (fig. 1).

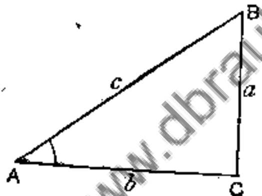


Fig. 1

The side AB is called the hypotenuse, and regarding the remaining sides from the angle A, BC is the opposite side and AC the adjacent side.

The ratio, $\frac{\text{opposite side}}{\text{hypotenuse}}$, is called the **sine** of the angle from which the triangle is regarded.

E.g. sine A (usually written sin A) $= \frac{BC}{AB}$ or $\frac{a}{c}$.

The ratio, $\frac{\text{adjacent side}}{\text{hypotenuse}}$, is called the **cosine** of the angle.

E.g. cosine A (briefly written cos A) $= \frac{AC}{AB}$ or $\frac{b}{c}$.

The ratio, $\frac{\text{opposite side}}{\text{adjacent side}}$, is called the **tangent** of the angle.

E.g. tangent A (briefly written tan A) $= \frac{BC}{AC}$ or $\frac{a}{b}$.

You will notice that the sides are lettered according to the angle to which they are opposite, but in small letters.

* The originator of the subject is said to be Hipparchus (160 B.C.).

EXERCISE XII (A)

1. Measuring the sides of the triangle constructed, find sin 40°, cos 40° and tan 40°.
2. The remaining acute angle is 50°. Regarding the sides from this angle, find sin 50°, cos 50° and tan 50°.
3. What conclusion do you draw concerning the ratios of an angle and the ratios of its complement?
4. Construct appropriate right-angled triangles, take measurements, and find the trigonometrical ratios of 30°, 45°, 60°, 80°, and 10°.
5. If one of the remaining angles of a right-angled triangle is very nearly 90°, why is the sine of that angle very nearly 1, and the cosine very nearly 0? Similarly explain why the sine of a very small angle is nearly 0 and the cosine nearly 1.

Remember that the terms sine, cosine and tangent merely denote ratios, and may be regarded as algebraic numbers.

The ratios for angles up to 90° will be found at the end and in the book of tables referred to in Chapter XVII, **13**.

2. Simple Applications of the Trignometrical Ratios.

ABC is a right-angled triangle, with ∠BAC = 35°, ∠ACB = 90° and AC (b) 2·5 cm. (fig. 2). Find the remaining sides.

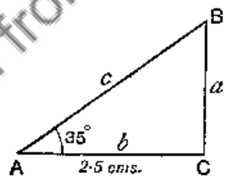


Fig. 2

$$\frac{a}{b} = \tan 35°, \quad \therefore a = b \tan 35° = 2{\cdot}5 \times {\cdot}7002 = 1{\cdot}75 \text{ cm.}$$

$$\cos 35° = \frac{b}{c}, \quad \therefore c = \frac{b}{\cos 35°} = \frac{2{\cdot}5}{{\cdot}8192} = 3{\cdot}05 \text{ cm.}$$

EXERCISE XII (B)

Find the remaining sides of the following right-angled triangles, C being the right angle:

1. A = 30°, a = 3 in. 2. B = 50°, a = 3·2 cm.

3. A = 30°, c = 3 in.
4. B = 50°, c = 3·2 cm.
5. A = 85°, b = 4 in.
6. A = 40°, c = 2·5 in.
7. Find the area of each of the above triangles.
8. If, in fig. 1, BC represents a vertical object and AC a horizontal line, ∠A is called the angle of elevation of the top of the object. When AC and ∠A are known, BC can be calculated.

   The elevation of the top of a tower at a point 300 ft. from its foot is 40°. Calculate the height of the tower.
9. If the sun-shadow cast by a vertical pole 6 ft. high is 8 ft., calculate the altitude of the sun.
10. The angle that BA makes with the horizontal through B (fig. 1) is called the angle of depression. From the property of parallels, the angle of depression of BA is equal to the angle of elevation of AB.

    From the top of a cliff 500 ft. high, the angles of depression of two boats at sea are observed to be 45° and 30° respectively; the line joining the boats points directly to the foot of the cliff. Find the distance between the boats.

3. The Length of any Parallel of Latitude.

We shall assume the earth to be a sphere.

Let ∠POW = the angle of latitude (L) of any place P (fig. 3).

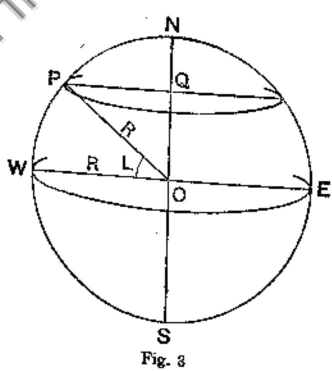


Fig. 3

PQ is the radius of the line of latitude of P, and OP the radius of the earth.

Then ∠QPO = alt. ∠POW = L.

Hence, regarding $\triangle PQO$ from $\angle QPO$,

$$\frac{PQ}{OP} = \cos L;$$

$$\therefore \; PQ = OP \cdot \cos L = R \cos L.$$

Now, since the line of latitude of P is the circumference of the circle of which PQ is the radius, we have:

$$\text{Length of parallel of latitude} = 2\pi \times PQ$$
$$= 2\pi R \cos L.$$

The portion of this line of latitude lying between two lines of longitude, one degree apart, is $\frac{2\pi R \cos L}{360}$, since the complete cycle is 360°.

At the Equator, L = 0°, *and since* cos 0° = 1, *this formula becomes* $\frac{2\pi R}{360}$.

Example.—Taking the earth to be a sphere of 4000 miles radius, find the length of the line of latitude 51° N. between any two lines of longitude one degree apart.

From the tables, cos 51° = ·6293.

$$\text{Required answer} = \frac{2\pi \times 4000 \times \cdot 6293}{360} = 43{\cdot}94 \text{ miles.}$$

It will be readily understood that the portion of a line of longitude lying between two lines of latitude one degree apart is the same for all longitudes and latitudes, and is equal to

$$\frac{2\pi R}{360} = \frac{\pi R}{180} = 69{\cdot}8 \text{ miles (approx.).}$$

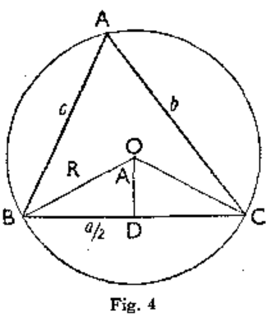


Fig. 4

4. The following relation between the sides and angles of a triangle is important, and very useful.

Let ABC (fig. 4) be the triangle considered.

Circumscribe a circle about the triangle ABC, O being the centre.

Join OB, OC and bisect $\angle$BOC by OD.

Then since $$\angle BOC = 2A$$
$$\angle BOD = A.$$

Also since $\triangle ODB = \triangle ODC$ in all respects, $\angle$ODB is a right angle and $BD = \frac{a}{2}$.

Now $$\sin A = \frac{a}{2}\Big/R = \frac{a}{2R}.$$

$$\therefore\ 2R = \frac{a}{\sin A}.$$

Similarly, $$2R = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

and each fraction = 2R.

Since $$\frac{a}{b} = \frac{\sin A}{\sin B},\ \text{and}\ \frac{b}{c} = \frac{\sin B}{\sin C},$$

*the ratio of any two sides is equal to the ratio of the sines of the opposite angles.**

5. Application.

When two angles and one side of a triangle are given, the remaining parts can be calculated.

EXAMPLE.†—To find the position of an inaccessible object C, the following measurements were made at a base line AB (fig. 5).

$$AB = 120\ \text{yd.},\ \angle BAC = 35°,\ \angle ABC = 60°.$$

The angle $ACB = 180° - (35° + 60°) = 85°$.

* In the above we have assumed that all the angles are acute, and in fact we have not yet defined the sine of an obtuse angle. Later we give the definition:

$$\sin(180° - A) = \sin A.$$

(See Chap. XXIII, p. 271.)

It can then be easily proved that the relations just given are true even if one angle is obtuse.

† Method used by Thales (600 B.C.) to find the distance of a ship from the seashore.

To find AC and BC, we have:

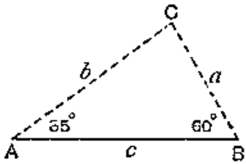


Fig. 5

$$\frac{a}{c} = \frac{\sin 35^\circ}{\sin 85^\circ};$$

$$\therefore \quad a = c\,\frac{\sin 35^\circ}{\sin 85^\circ}$$

$$= 120 \times \frac{0{\cdot}5736}{0{\cdot}9962} \text{ yd.}$$

$$= 69{\cdot}1 \text{ yd.}$$

From the relation $\frac{b}{c} = \frac{\sin B}{\sin C}$, b can be found in like manner.

6. Circular Measure of Angles.

The curved length of the arc between two radii of a circle is proportional to the angle between them. It is also proportional to the length of the radius. This relation between arc, radius and angle is often of great use in the measurement of angles.

The unit angle in circular measure is the angle for which the arc is equal in length (measured along the curve) to the radius (fig. 6). It is called the RADIAN.

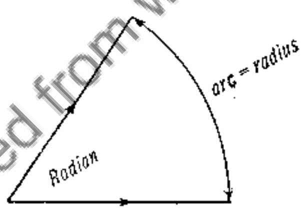


Fig. 6

For an angle of 2 radians the arc is twice the length of the radius, and so on. Thus the relation is:

$$\text{Arc} = \text{radians} \times \text{radius},$$

or

$$\text{Radians} = \frac{\text{arc}}{\text{radius}}.$$

It is evident that the number of radians in a straight angle is the same as the number of times the length of the radius is contained in the length of the arc of a semicircle of that radius, namely π, which is approximately $3\frac{1}{7}$.

This enables us to calculate approximately the size of the radian in degrees.

$$3\tfrac{1}{7} \text{ radians} = 180°, \therefore 1 \text{ radian} = \frac{180°}{3\frac{1}{7}} = 57\tfrac{3}{11}° \text{ approx.}$$

You will no doubt understand why it is less than the angle of an equilateral triangle.

Construct a radian as accurately as you can by laying a length of thread or thin wire equal to the radius along the arc of a circle. Measure the angle by protractor.

EXAMPLES.

(i) Convert 105° into radians and find the length of arc subtended at a radius of 12 in.

$$\text{Radians} = \frac{105}{57\frac{3}{11}} = \frac{105 \times 11}{630} = 1\tfrac{5}{6}.$$

$$\text{Arc} = \text{radians} \times \text{radius} = 1\tfrac{5}{6} \times 12 \text{ in.} = 22 \text{ in.}$$

(ii) Find the angle subtended by a circular arc of 25 cm. at a radius of 10 cm.

$$\text{Radians} = \frac{\text{arc}}{\text{radius}} = \frac{25}{10} = 2\tfrac{1}{2}, \text{ or } 2\tfrac{1}{2} \times 57\tfrac{3}{11}° = 143\tfrac{2}{11}°.$$

EXERCISE XII (c)

1. Convert the following angles into radians:
   20°, 30°, 45°, 60°, 90°, 130°, 200°, 270°, 310°, 360°.
2. Express the following radians in degrees:
   $$0{\cdot}2,\ 1{\cdot}1,\ 2{\cdot}4,\ 3{\cdot}14,\ 5{\cdot}3,\ 6{\cdot}28,\ 2\pi,\ \frac{3\pi}{2}.$$
3. Find the arcs limited by the angles in Exercises 1 and 2 at a radius of (a) 5 cm.; (b) 10 in.
4. Draw a circle of radius 1 in. By drawing radii divide it into six equal sectors. Calculate the length of each of the six arcs.

7. Area of a Sector of a Circle.

A circle can be divided into a number of equal sectors by drawing radii which divide the angle at the centre (i.e. a cycle or 360°) into the same number of equal parts. For example, six radii, drawn at $\frac{360°}{6} = 60°$, divide the circle into six equal sectors.

The circumference is divided into equal arcs also.

The area of a sector is the same fraction of the circle that its angle is of a cycle (360° or 2π radians) and that its arc is of the whole circumference.

If R is the radius and $n°$ the angle of the sector, the area is $\frac{n}{360}$ of πR^2. If the angle is measured in radians and is A radians, then the area is $\frac{A}{2\pi}$ of πR^2, which equals $\frac{1}{2}AR^2$.

If a is the arc of the sector (fig. 7), the fraction it is of the circumference is $\frac{a}{2\pi R}$;

$$\therefore \text{ area of sector} = \frac{a}{2\pi R} \times \pi R^2 = \tfrac{1}{2}aR.$$

The last two results agree, since $a = AR$.

It will be seen that $\frac{1}{2}aR$ is the same as the formula for the area of a triangle if a is regarded as the base and R as the altitude.

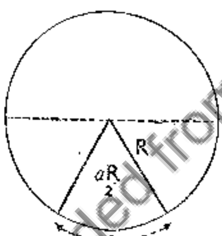


Fig. 7

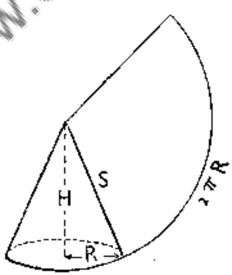


Fig. 8

8. Curved Surface of a Cone.

If the curved surface of a cone could be peeled off and laid out flat as indicated in fig. 8, it would have the shape of a sector of a circle, the radius being equal to the slant height (S) of the cone and the arc to the circumference ($2\pi R$) of the base of the cone.

The area of this sector is $\frac{2\pi RS}{2} = \pi RS$.

If H is the altitude of the cone, $S = \sqrt{H^2 + R^2}$.

Exercise XII (d)

1. If the arc of a sector of a circle is 16″ and its radius 12″, what is its area?
2. What is the angle of the sector of Ex. 1 and by how much does the direction of the arc change from one end of it to the other?
3. The length of a circular curve of railway is 300 yd. and the change of direction 16°. Calculate the radius of the curve.
4. The angle of a sector is 160° and its arc 20 cm.; find its radius and area.
5. A figure is bounded by two concentric arcs of length 12 and 8 cm., and the parts of two radii, each part 4 cm. long. Find the area of the figure, and the length of the full radius.
6. Find the area of the curved surface of a cone 4 in. high and radius of base 3 in.

Exercise XII (e)

1. Complete the working of the example under fig. 5, and find also the perpendicular distance from C to the base line.

Employing relation $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, calculate the remaining sides of the following triangles:

2. $A = 60°$, $B = 80°$, $a = 2$ cm.
3. $A = 60°$, $B = 80°$, $b = 2$ cm.
4. $A = 60°$, $B = 80°$, $c = 2$ cm.
5. $B = 50°$, $C = 40°$, $a = 10$ cm.
6. $A = 100°$, $C = 30°$, $b = 6$ in.
7. What is the diameter of the circum-circle of each of the triangles of Exercises 2 to 6?
8. (i) The directions of an object make angles of 60° and 50° with the directions of a base line 1000 yd. long, when viewed from the ends of the base line. Find the position of the object.

   (ii) If the object is a pyramid the elevation of the top of which is 11° when sighted from the end of the base line at which the direction is 60°, find its height, and its elevation if sighted from the other end of the base line.

9. Show that the area of a triangle is

$$\frac{bc}{2}\sin A \quad \text{or} \quad \frac{ab}{2}\sin C \quad \text{or} \quad \frac{ac}{2}\sin B.$$

10. Calculate the areas of the triangles in Exercises 2 to 6.

11. Assuming the earth to be a sphere of 8000 miles diameter, what is the circumference of the circle of latitude 52°?

The earth makes one revolution in 24 hr. (approximately); what is the speed at latitude 52° in miles per hr?

12. Explain, with a diagram, how you would find the height of a tree, if you have a set-square whose angles are 30° and 60°.

13. A tower, 30 ft. high, is surmounted by a vertical flagstaff. At a point P 40 yd. from, and in the horizontal plane through the foot of the tower, the flagstaff and the tower subtend equal angles; what is the length of the flagstaff?

14. A right-angled triangular field has one side 150 yd. long, the angle opposite it being 58°. Find the hypotenuse.

15. South America has roughly the shape of two triangles with a common base, as shown in fig. 9.

From the data given, calculate the approximate area of the country.

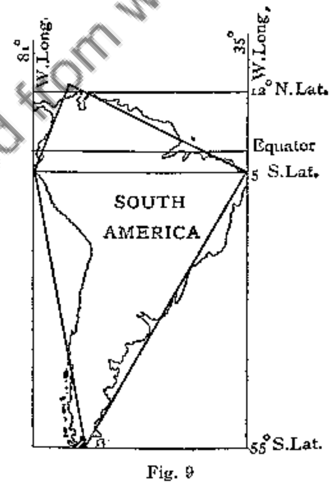


Fig. 9

16. Refer to your atlas and you will see that India also has the shape of two triangles placed as in fig. 9. In this case

the base is on the 25° N. latitude line, the ends being respectively at 67° and 93° E. longitude, and the northern and southern vertices respectively at 34° and 8° N. latitude.

Determine the approximate area of the country.

17. Find the distance between New York (41° N., 74° W.) and Madrid (41° N., $3\frac{1}{2}$° W.), measured along the line of latitude.

18. A ship sailing from Portsmouth ($50\frac{1}{2}$° N., 1° W.) to New York (41° N., 74° W.) sails along the meridian until a little south of latitude 50° N. is reached, and then sails west until its longitude is 44° W., after which it sails south until its latitude is 41° N., when it resumes its westerly course for the remainder of the voyage. Calculate the approximate length of the voyage.

19. Fig. 10 represents two pulleys connected by a taut belt.

The diameters of the pulleys are 3 ft. and 2 ft. respectively, and their centres are 5 ft. apart. Find by trigonometry:

(i) The angle TOQ.

(ii) The length of TR.

(iii) The reflex angle TOT_1.

(iv) The angle RQR_1.

(v) The length of the belt.

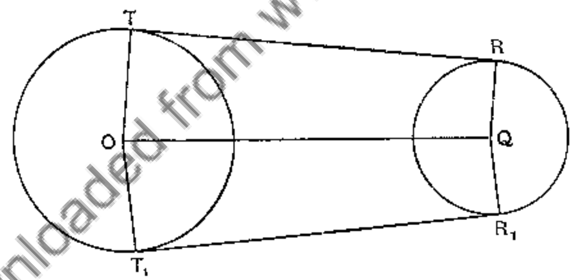


Fig. 10

Check your result by drawing and measurement.

If the larger pulley makes 100 revs. per minute, find:

(i) The speed of the other pulley.

(ii) The speed of the belt in feet per second.

20. A disc of diameter 6 in. is hung against a wall by a string which passes over a nail 5 in. above the centre of the disc, and round a part of the rim of the disc.

Sketch the arrangement, and neglecting the thickness of the nail, calculate the length of the string.

REVISION EXERCISE 1

1. Simplify:

(i) $12 - 7 + (-8) - 4 - (-3)$.

(ii) $2 \times -3 + (5)^2 - 4 \div -2 - 3(6-4) - (-3)^2$.

(iii) $6(a+b) - 2(a+b) + 4(a+b) - (a+b)$.

(iv) $5a(a-b) + 2(x+y) + 2a(a-b) - 4(x+y) - 3a(a-b)$.

2. (i) Subtract 12 from -8. (ii) From -7 subtract -13.

(iii) Subtract $2x^2 - 5x - 8$ from $-6x^2 + 3x - 5$.

(iv) From $-3(2a - 6b)$ take $-7(a - 3b)$.

3. Calculate the position of the point midway between the points situated at distances 3·2 and $-1·4$ in. respectively from zero.

4. Find the value of:

(i) $(3x + 2y)a - b(2x - 3y)$, when $ax = 3$, $ay = -2$, $bx = 1$, and $by = 1$.

(ii) $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, when $a = 4$, $b = -5$, $c = 1$, and when $a = 2$, $b = 2$, $c = -4$.

5. A kite has the shape of an equilateral triangle with a semicircle on one side. If s is the length of the side of the triangle, find (i) the perimeter, (ii) the area of one face of the kite.

6. If $y = 367 + 2·35\,(x - 36)$, find the difference between the values of y when $x = 52$ and $x = 12$.

7. Show that $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$. From this identity write down the square of $(a + 2x - 3y)$.

8. For what value of x does $\frac{8x}{3+x}$ equal (i) 7, (ii) 9? Can $\frac{8x}{3+x}$ have the value 8?

9. Solve (i) $9(x+3)^2 + 15(x+5)^2 = 24(x+8)^2$.

(ii) $\frac{5(7x+6)}{23} + \frac{9x-1}{2} = 50 + \frac{2x-3}{3}$.

10. If $\frac{M}{N} = \frac{1+x}{1-x}$, show that $x = \frac{M-N}{M+N}$.

11. The perimeter of a 60°, 30°, right-angled triangle is 30 in.; find the length of each side, and the area of the triangle.

12. Find (i) $\frac{\sin 2x - \cos 2x}{2}$, when $x = 30°$, and when $x = 45°$.

(ii) $\frac{\sin \frac{D+A}{2}}{\sin \frac{A}{2}}$, when $D = 48°$ and $A = 60°$.

13. Draw a perpendicular to the side opposite the angle C of $\triangle ABC$, and show that $c = a \cos B + b \cos A$. Find similar equations for a and b also.

14. On each half of the diameter of a semicircle describe a semicircle. Show that the outer arc is equal to the sum of the two inner arcs. Repeat the construction with each of these two semicircles. The outer arc is equal to the sum of the four arcs now obtained.

If the process is continued, what line is ultimately reached and what obviously wrong conclusion seems to follow? (A paradox! a paradox! a most ingenious paradox!)

Can you explain the fallacy?

CHAPTER XIII

GRAPHS *

1. Graphs of Simple Expressions.

Previous work (p. 56) suggests that in representing values graphically, not only should positive values be shown, but negative values also.

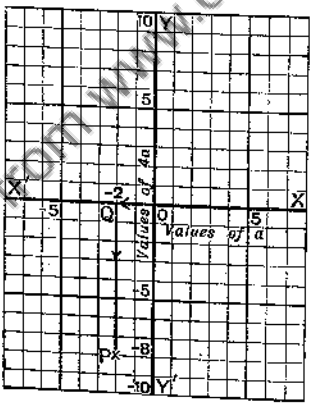


Fig. 1

If, in fig. 13, Chapter V, **21**, the horizontal line be extended to the left, and measurements made below the horizontal line, negative values of a and of $4a$ can be represented.

The usual method is to draw two straight lines X′OX and Y′OY at right angles, as in fig. 1. These lines are called *axes*,

* The pioneers of graphs were the Frenchmen Descartes (1596–1650) and Fermat (1601–1665).

and their intersection, the *origin*. Values of a are measured along, or parallel to, the horizontal axis, and values of $4a$ along, or parallel to, the vertical axis. E.g. when a is -2, $4a$ is -8.

These values are represented as follows:

Move from the origin 0 to -2, on the left, and then move downwards and parallel to 0Y′, through a distance of -8, as shown by the scale on the vertical axis. Call the final position P. Then P indicates the two values -2 and -8. The distances 0Q and QP are called the co-ordinates of P, 0Q being named the abscissa (plural, abscissae) and QP the ordinate. In stating the co-ordinates it is usual to give the abscissa first; thus the co-ordinates of P are -2, -8.

You must not conclude that ordinates are always negative when the corresponding abscissae are negative.

It is common practice to use x as the symbol. Our object now is to examine the changes in the value of various expressions containing x when the value of x is changed.

Take the simple expression $2x$, and tabulate, as below, its value when x is given the various values shown.

Value of x	-4	-3	-2	-1	$=x=$	0	1	2	3	4	5
Value of $2x$	-8	-6	-4	-2	$=2x=$	0	2	4	6	8	10

Examining these numbers, it is seen that:

(i) The values of x increase by equal amounts, viz. 1.

(ii) The values of $2x$ increase by equal amounts, viz. 2.

It is clear that when equal changes are made in the value of x, the corresponding changes in $2x$ also are equal, but not necessarily equal to the changes in x.

Thus, when x changes from 2 to 3, an increase of 1,
$2x$ changes from 4 to 6, an increase of 2,
and when x changes from -4 to -3, an increase of 1,
$2x$ changes from -8 to -6, an increase of 2.

In this case the change in the expression is twice that in x. It is observed also that:

(i) When x is $+$, $2x$ is $+$.
(ii) When x is $-$, $2x$ is $-$.
(iii) When x is 0, $2x$ is 0.

Now plot the values, as in fig. 2, and join the points by straight lines. What do you find?

It is not difficult to prove that all the points are in one straight line.

The *Graph* of $2x$ is thus a *straight line.*

Produce the graph in both directions. Take values of x for which you have not calculated the values of $2x$, say $x = \frac{1}{2}, -\frac{1}{2}, -5$, etc., and from the graph read off the corresponding values of $2x$. Check these values by actual calculation, and you will find that the graph gives correct results.

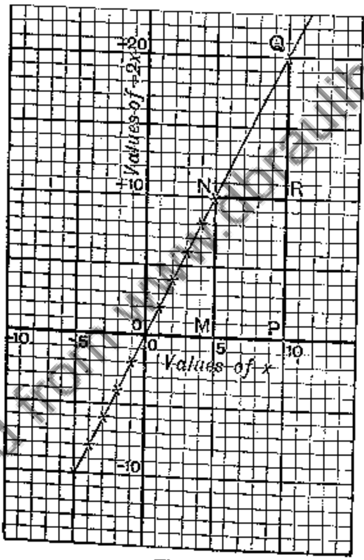


Fig. 2

Observe further that:

(i) The graph could have been drawn if two points only had been plotted.

(ii) The graph passes through the origin (0, 0).

(iii) Regarded from the origin towards the right, the graph has an up gradient.

(iv) The ratio of the length of any ordinate to its horizontal distance from the point at which the graph cuts the axis of x, is constant.

$$\text{E.g. } \frac{PQ}{OP} = \frac{MN}{OM}.$$

(v) The ratios referred to in (iv) are each equal to the ratio $\frac{RQ}{NR}$, where NR is parallel to the axis of x, i.e. to the ratio

$$\frac{\text{Difference between any two ordinates}}{\text{Difference between the two corresponding abscissae}}$$

The ratios named in (iv) and (v) are very important, for they measure the gradient of the graph.

You have probably recognized these ratios as the *tangent of the angle the graph makes with the axis of x*.

Thus: $$\frac{PQ}{OP} = \tan \angle QOP.$$

In this case $\frac{PQ}{OP} = \frac{20}{10} = 2$, hence $\tan \angle QOP = 2$.

The graph being a straight line, its gradient is, of course, constant throughout its length.

2. It is now proposed to determine upon what the gradient depends.

Examine in the same manner the following expressions, namely, $3x$, $4x$ and $\frac{1}{2}x$. Draw the graph of each expression on the same axes as those used for the graph of $2x$ (fig. 3).

Compare the graphs, and observe that:

(i) All pass through the origin.

(ii) All have up gradients.

(iii) The greater the coefficient of x, the greater is the gradient. It is evident that since the expressions differ only in coefficients, the gradient depends upon the coefficient, and may be said to be equal to it. From a table of tangents, find the angle the graph makes with the axis of x in each case. Verify by measurement.*

Now plot the graph of $-2x$, and contrast it with that of $+2x$ (fig. 3).

It will be at once observed that the graph of $-2x$ has a down gradient, and you have doubtless concluded that this change in the kind of gradient is due to the change in sign of the coefficient.

A positive coefficient of x gives an up gradient to the right; a negative coefficient a down gradient to the right.

* The scales of both axes must be the same, otherwise the measured angle will not agree with that given in the table of tangents.

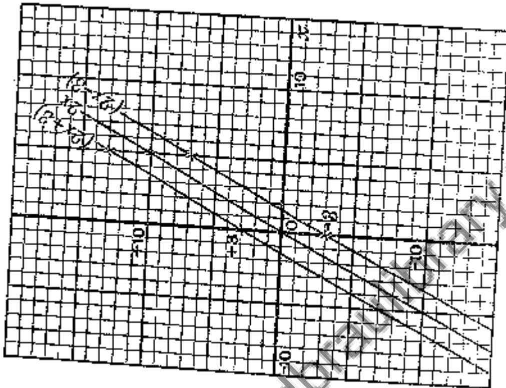


Fig. 4

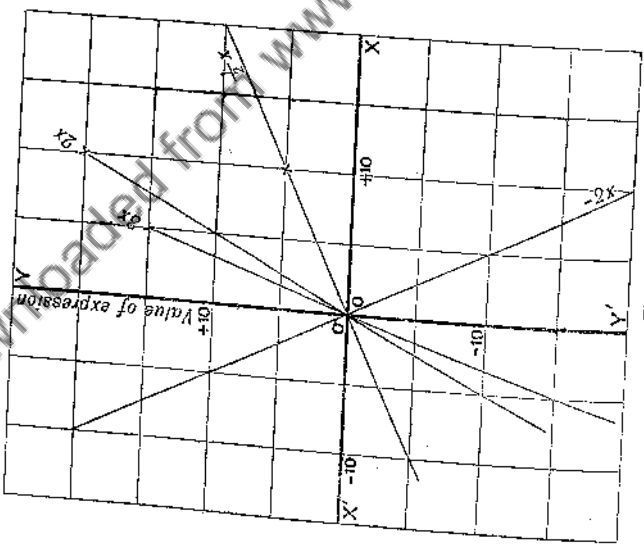


Fig. 3

3. In order to examine the effect upon the graph, of adding to, or subtracting from $2x$ a constant number, say 3, plot the graphs of $2x + 3$ and $2x - 3$, and contrast them with the graph of $2x$ (fig. 4). It will be observed that:

(i) All the graphs have the same gradient.

(ii) The graph of $2x + 3$ cuts the vertical axis at a distance 3 above the origin.

(iii) The graph of $2x - 3$ cuts the vertical axis at a distance 3 below the origin.

Similarly, examine the graphs of $-2x + 3$ and $-2x - 3$.

Consider now the point at which, say, the graph of $2x - 3$ cuts the axis of x.

At this point the value of the expression is 0, and the corresponding value of x is seen to be $1\frac{1}{2}$. This value of x may be obtained by solving the very simple equation

$$2x - 3 = 0.$$

We have, then, a means of determining the point of intersection of the graph with the axis of x.

4. Summary.

(i) The graph of an expression of the type $ax + b$, in which a and b are constant numbers, is a straight line.

(ii) The gradient of the graph depends upon the coefficient of x, and is "up" if the coefficient is positive, "down" if negative.

(iii) The position of the graph with respect to the origin depends upon the added constant. If positive, the graph cuts the vertical axis above the origin; if negative, below the origin.

It is usual to call the value of the expression y, and the axis upon which it is shown, the axis of y.

The equation $$y = ax + b$$

is called a linear equation, for its graph is a straight line. Observe that it contains the first power only of x.

y is said to be a linear function of x.

Notice that when a value is given to x, the value of y becomes definite.

The various forms that the graph of $y = ax + b$ may take are shown in fig. 5.

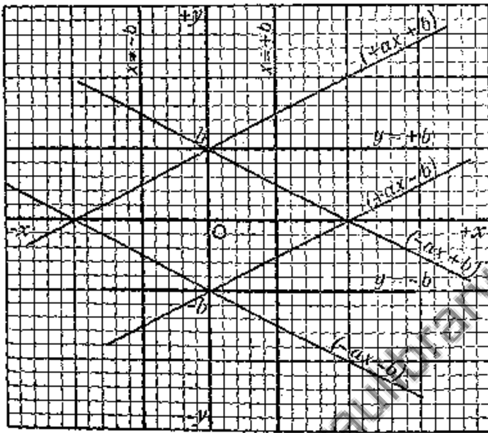


Fig. 5

Exercise XIII (a)

1. Without drawing the graphs, compare the gradients of the graphs of the following expressions, and state where each graph will cut the axes of y and x:

   (i) $3x - 5$. (ii) $-x + 4$. (iii) $\frac{2}{3}x - 7$.

2. Write down the equations of graphs which have the following properties.

   (i) Gradient $+3$, intersects the axis of y at 7 below the origin.

   (ii) A down gradient of 3, intersects the axis of y at 5 above the origin.

   (iii) An up gradient of 5, intersects the axis of y at 3 below the origin.

   (iv) Gradient -2, intersects the axis of x at $+6$.

   (v) Gradient $2\frac{1}{2}$, passes through the origin.

3. A graph is parallel to the axis of x and intersects the axis of y at $+5$. What is its equation?

4. Write down the equation to each of the graphs in fig. 5.

5. Compare the graphs of the following expressions:

   $$3x - 5, \quad 3x + 2 \quad \text{and} \quad 2x - 5.$$

6. What changes are made in a graph when the expression is doubled?

7. Plot the graph of the equation $x = 2y + 3$.
What change occurs when the coefficient of y is reduced until it becomes 0?
8. Write down the equation to the graph obtained in Exercise V (F), No. 2, and also to those in figs. 9 and 11, Chapter V.

5. Intersection of Graphs.

Since a point on the graph of an expression gives the value of the expression for that particular value of x, it follows that, at the point of intersection of two graphs, the expressions which they represent must have the same value.

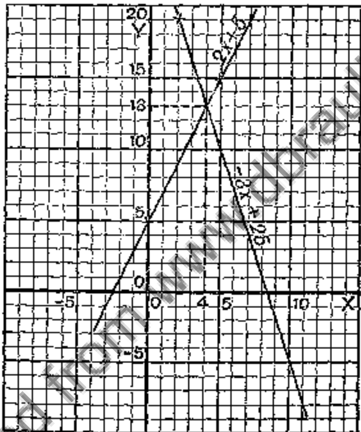


Fig. 6

Moreover, the x co-ordinate of the point of intersection must be the value of x for which the expressions have the same value, that is, are equal.

Draw the graphs of $2x + 5$ and $-3x + 25$ on the same axes, and verify this (fig. 6).

The value of x for which two expressions are equal may be found quickly by equating the expressions and solving the equation. Thus, taking the above expressions:

$$2x + 5 = -3x + 25,$$
$$2x + 3x = 25 - 5,$$
$$x = 4.$$

For this value of x, $\quad 2x + 5 = 13,$
and
$$-3x + 25 = 13.$$

6. Interpolation and Extrapolation.

When the value of a function of, say, x is determined for a value of x between values for which the values of the function are already known, the process is called *Interpolation.*

On the other hand, when the value of x is not between values of x for which the values of the function are known, the process is called *Extrapolation.* Values determined by interpolation or by extrapolation are not necessarily correct.

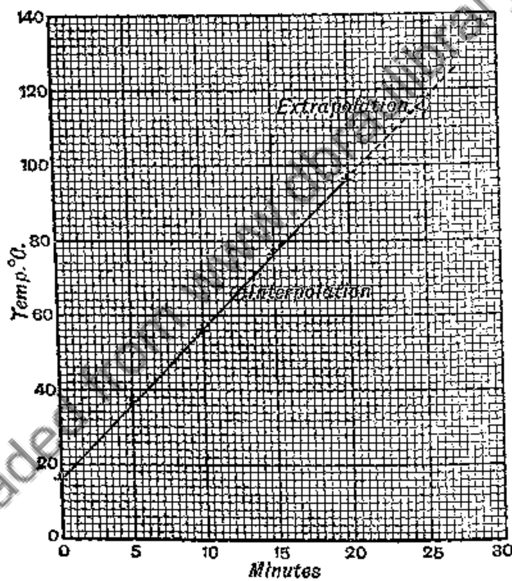


Fig. 7

For example, the graph given in fig. 7 shows the temperature of a quantity of water when heated by electrical means for the time shown. Readings were taken every 5 min. for 20 min.

If the graph is produced, the temperature indicated by it corresponding to a period of 25 min. is 116° C. But this is incorrect, for water boils at 100° C., and the temperature does not rise above this.

The correctness of a result obtained by extrapolation or interpolation depends upon whether the function is continuous throughout values which include those of the point under consideration.

Note.—In drawing a graph it is not necessary to adopt always the same scale on both axes or to number the point of intersection of the axes, 0, 0.

Exercise XIII (b)

1. Why must the expressions of graphs which intersect have unequal coefficients of x?
2. Write down the expressions of two graphs which will not intersect.
3. State whether the graphs of $-2x + 5$ and $2x - 6$ will intersect, and if so, find the co-ordinates of the point of intersection.

Find graphically the values of x for which the following functions have the same value. Check by calculation.

4. $2x + 3$ and $4x - 3$.
5. $\frac{3}{2}x + 2$ and $3x - 1$.
6. $4 - 3x$ and $2(x + 8)$.
7. $4x - 18$ and $-(3x + 10)$.
8. Draw a graph showing the cost of articles to 1000 at 3d. each. Show the effect of a reduction at the rate of 2/6 per 100 after the first 100.
9. Solve by a graphic method the following question:

   A and B are approaching each other on the same road, A walking at 5 miles an hour and B cycling at 9 miles an hour. If B is at the second milestone when A is at the twentieth, to which milestone will they be nearest when they meet?
10. Solve by a graphic method the following problem:

    A cyclist A is riding on a road out of a certain town at a rate of 8 miles per hour. A second cyclist B rides out on the same road at a rate of 10 miles an hour. If B passes the first milestone ten minutes later than A, which will be the milestone nearest to them when B overtakes A?
11. One clock, A, gains and another, B, loses uniformly. At noon on Monday, A is 30 min. slow and B 50 min. fast. At noon on the following Friday, A is 10 min. fast, and B 10 min. slow. Represent days on the axis of x and minutes fast and slow on the axis of y, and find graphically (i) the day and actual time at which the clocks indicated the same time, (ii) what that indicated time was.

Write down the equations to the graphs, and check you results by Algebra.

12. Plot the points $x = 1$, $y = 5$ and $x = -4$, $y = -5$. Join them by a straight line, and determine its equation.

13. Write down the equation to the straight line which cuts the axis of x at a distance 4 and the axis of y at a distance -6 from the origin.

14. Determine the equation to the straight line which passes through the points $x = 1$, $y = 1$ and $x = -3$, $y = 9$.

15. What is the equation to the straight line at right angles to that of the last question, and passing through the origin?

16. Calculate the distance between the points

$$x = 2, y = 5 \quad \text{and} \quad x = 6, y = 8.$$

17. Plot the point $x = -2y = 6$ and the point $y = -\frac{3}{2}x = 9$, and find the equation to the straight line joining them.

18. If $y = 3x + 5$, find the changes in y when x changes from 1 to 2, from -2 to $+2$, and from 3 to 0.

19. If $y = 8 - 3x$, find the ratio of the change in y to the change in x for each of the following changes in x, viz.: (i) 1 to 2; (ii) -1 to -3; (iii) 4 to -2; (iv) 2 to 2·001; (v) your own choice; write down your conclusions.

7. Applications.

Being able to write down the equation to a given straight-line graph, you are now in a position to understand its use in Science and Mathematics.

Example i.—The following numbers were obtained when a Fahrenheit and a Centigrade thermometer were used to determine, at various times, the temperature of a quantity of water which was being heated. The thermometers were read simultaneously.

Centigrade	15	20	25	30	40	60	80	100
Fahrenheit	59	68	77	86	104	140	176	212

Represent the Centigrade readings on the axis of x and the Fahrenheit on the axis of y, and plot points which have as co-ordinates these simultaneous values (fig. 8).

The points lie almost on a straight line. (Any deviation may be due to careless reading or imperfections of the thermometers.)

Draw the straight line which passes evenly between the points, and find its equation.

The interpretation is that Centigrade readings are converted into Fahrenheit by multiplying by 1·8 or $\frac{9}{5}$ and adding 32.

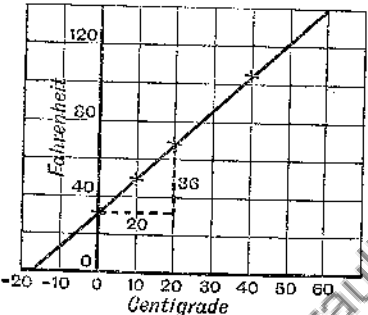


Fig. 8

Example ii.—The following numbers show the volume of a mass of gas when heated to different temperatures, the pressure being constant:

Temperature (°C.)	15	25	35	40	60
Volume in c.c.	150	155·25	160·5	163	173

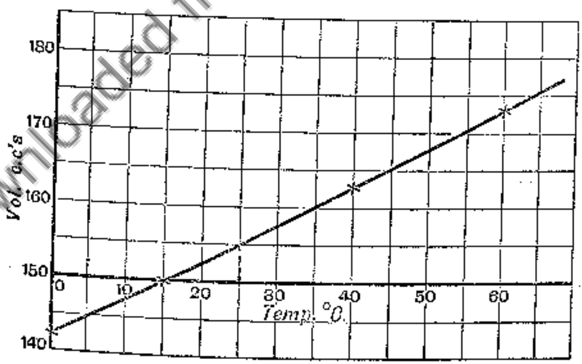


Fig. 9

Find the equation, and from it determine the temperature at which the volume of the gas would be zero (fig. 9).

The equation found being $y = 0{\cdot}52x + 142$,

we have $0 = 0{\cdot}52x + 142$,

from which $x = -273{\cdot}3$,

i.e. 273·3 degrees below 0° C.

EXAMPLE iii.—The table gives the total heat in a pound of steam at different temperatures:

Temperature (°C.)	80	100	110	120	130	150
Total heat units	631	637	640	643	646	652

Find the law connecting total heat and temperature.

You will find it inconvenient to make the intersection of the axes 0 for either axis (fig. 10).

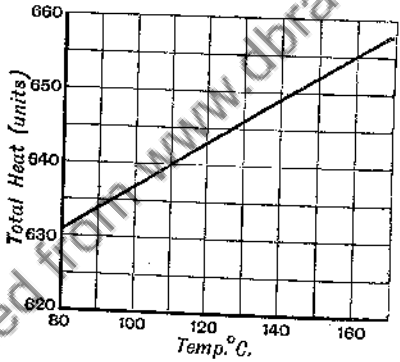


Fig. 10

The added constant can be found as follows:

The gradient will be found to be 0·3 (approx.).

Let b represent the constant; then

$$y = 0{\cdot}3x + b.$$

Take two known values of x and y, and solve the equation for b. Thus:

$$637 = (0{\cdot}3 \times 100) + b,$$

from which $b = 607$.

Hence the relation is $y = 0{\cdot}3\,x + 607$.

The accepted relation is $y = 0{\cdot}305x + 606{\cdot}5$.

Exercise XIII (c)

1. The force to raise a roller up an inclined plane, the height of which was varied, was found to be as follows:

Height (cm.)	0	10	20	30	35
Force (gm.)	8	16	24	32	36

Find the law connecting height and force.

2. A body moves with a uniform velocity of 5 ft. per second.

Draw a graph showing the distance covered in various intervals of time.

This graph is called the graph of positions.

Observe that the gradient is equal to the value of the velocity.

3. Using a set of pulleys, the force required to lift different weights was found to be as stated below:

Weight (lb.)	0	8	12	20	52
Force (lb.)	1·5	3·2	4·1	5·9	12·8

Find the equation connecting weight and force.

4. The following were the readings of a barometer when lowered into water:

Depth (in.)	0	1	2	5	8	10	15	20	25	30
Reading (in.)	30·0	30·07	30·15	30·37	30·59	30·7	31·07	31·46	31·83	32·2

Find the relation between the reading and the depth.

5. The following temperatures were taken every minute when a quantity of water was heated by a flame of constant power.

Find the equation connecting the temperature and the time of heating.

Time (min.)	0	1	2	3	4	5	6	7	8
Temperature (°C.)	15	18	21	24	27	30	33	36	39

6. The latent heat of steam at different temperatures is given in the following table. Find the law.

Temperature (°C.)	100	120	140	160	180	200
Latent heat	537	523	509	495·8	481	468

7. The table below gives the resistance of a length of platinum wire when its temperature is varied. Establish the equation connecting resistance and temperature.

Temp. (°C.)	15	20	40	60	80	100	120	150
Resistance (ohms)	105·2	106·9	113·8	120·7	127·6	134·5	141·3	151·7

8. In an experiment to determine the coefficient of expansion of benzene, the following numbers were obtained:

Temperature (°C.)	0	20	40	60	80
Volume	1·0	1·0241	1·0500	1·0776	1·1070

Plot these numbers, and find the equation connecting them.

8. Graph of Inverse Proportion.

Plot the graph of the expression $\frac{1}{x}$, i.e. of the equation:

$$y = \frac{1}{x}.$$

-3	-2	-1	$-\frac{1}{2}$	$=x=$	0	$\frac{1}{2}$	1	2	3	4	5
$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	$=\frac{1}{x}=$	$=\frac{1}{0}$* $=\infty$	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

* Any number divided by 0 is equal to infinity; no finite number is large enough to represent the result. Infinity is denoted by the symbol ∞.

On plotting the points, you find that there are two distinct graphs or branches; one in the first quadrant, and the other in the third (fig. 11).

Examine the graphs, and verify that they possess the following characteristics:

(i) Taking any two ordinates, and the corresponding abscissae, the ratio of the ordinates is equal to the inverse ratio of the corresponding abscissae. (This is why the graph is called the graph of Inverse Proportion.)

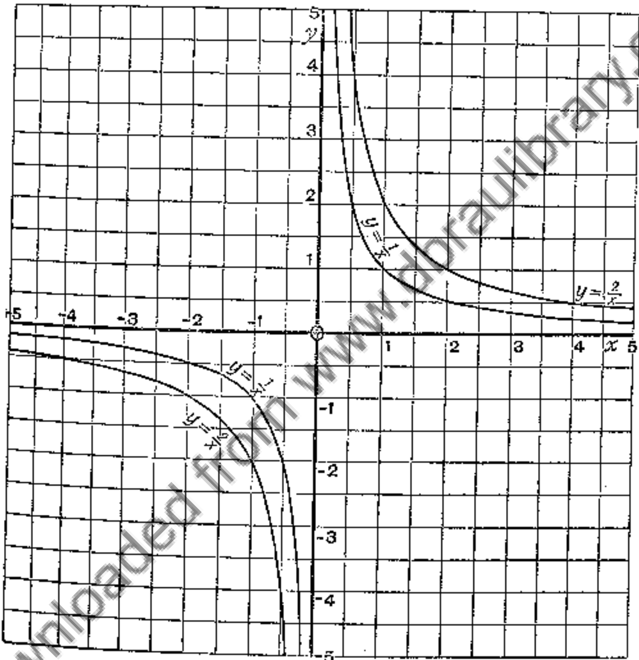


Fig. 11

(ii) Each curve has a bend, or elbow, opposite the origin.

(iii) On each side of the elbows, the curves get straighter and approach the axes, but never actually meet them. It is usual to say that the graphs cut the axes at infinity.

The complete graph, composed of the two branches, is called a **hyperbola.**

9. Plot now the graphs of the equations:

$$\text{(i) } y = \frac{2}{x}. \qquad \text{(ii) } y = \frac{2}{x} + 3. \qquad \text{(iii) } y = \frac{2}{x} - 3.$$

Comparing each of these graphs with those of $\frac{1}{x}$, it is seen that the effect of the 2 has been to move the graphs away from the axes (fig. 11), of the +3 to raise the graphs and of the −3 to lower the graphs. Graphs (ii) approach the straight line $y = +3$ and those of (iii) $y = -3$.

10. Applications.

If, when simultaneous values of two quantities are plotted, the graph appears to be like those on p. 155, the truth of the assumption that the law is $y = \frac{a}{x}$ * can be verified as follows:

If $y = \frac{a}{x}$, then, if z is written for $\frac{1}{x}$ (the reciprocal of x), $y = az$.

Hence, if y and z are plotted, the graph is a straight line of gradient a.

If the law is $y = \frac{a}{x} + b$, then, substituting as before, $y = az + b$, the graph of which is again a straight line of gradient a, but which cuts the axis of y at b.

Referring to the graph of $y = \frac{2}{x} + 3$, plot y and $\frac{1}{x}$, and verify the above statement.

-4	-3	-2	-1	$=x=$	0	1	2	4
$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	$=\frac{1}{x}=$	∞	1	$\frac{1}{2}$	$\frac{1}{4}$
$2\frac{1}{2}$	$2\frac{1}{3}$	2	1	$=y=$	∞	5	4	$3\frac{1}{2}$

You find that the graphs for positive and negative values of x

* Another way of expressing this relation is $xy = a$.

form a continuous straight-line graph which cuts the axis of y at 3, and has a gradient of 2 (fig. 12).

The equation is, therefore, $y = 2\left(\frac{1}{x}\right) + 3$, i.e. $y = \frac{2}{x} + 3$.

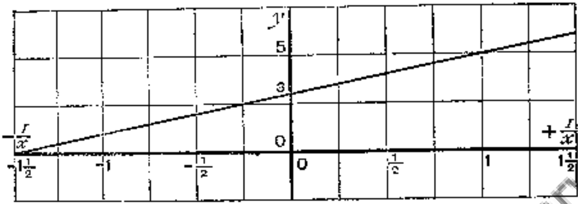


Fig. 12

Exercise XIII (d)

Construct graphs to show the following:

1. The number of rails of different lengths required for a mile of railway.
2. The number of revolutions made by wheels of different diameters in covering a fixed distance.
3. The speed of a moving body and the time it takes to pass over a fixed distance.
4. The following numbers were obtained in an experiment for ascertaining how the volume of a gas changes when the pressure is varied:

P.	10	15	20	25	30	45
V.	45	30	22	18	15	10

Find the relation between pressure and volume.

5. The table shows the distance from a fulcrum at which a given weight must be placed to give a certain leverage.

Weight (lb.)	10	20	40	50
Distance (in.)	50	25	12·5	10

Find the relation between weight and distance.

6. A graph of the form $\frac{a}{x} + b$ passes through the points $x = 3$, $y = 3$ and $x = -1$, $y = -1$. Find its exact equation, and the straight lines which it approaches.

CHAPTER XIV

SIMULTANEOUS SIMPLE EQUATIONS, LITERAL EQUATIONS, PROBLEMS

1. Simultaneous Simple Equations.

We have seen on p. 147 that it is possible for two otherwise different expressions to have equal values while the value of the unknown number is the same in both.

When the expressions form part of equations, the equations are called simultaneous equations.

Taking the expressions $3x - 4$ and $2x + 5$; if we call the value of each expression y, we can write the example in equational form, thus:

$$y = 3x - 4,$$
$$y = 2x + 5;$$

or, transposing terms, thus:

$$3x - y = 4,$$
$$2x - y = -5;$$

or we might have different multiples of the equations, thus:

$$9x - 3y = 12,$$
$$-4x + 2y = 10,$$

where the first equation has been multiplied by 3 and the second by -2. When given such a pair of equations, the object is to find the values of x and y for which each equation holds good. In other words, to find the values of x and y which, when substituted for these symbols, make the expressions on the left equal to 12 and 10 respectively.

There are several methods of solving these problems.

METHOD I. This has been already indicated on p. 147.

Find the value of either x or y in each equation, and equate the results.

$$9x - 3y = 12, \quad \ldots \quad \text{(i)}$$

$$-4x + 2y = 10. \quad \ldots \quad \text{(ii)}$$

From (i), $$-3y = -9x + 12,$$

$$y = \frac{-9x + 12}{-3},$$

i.e. $$y = 3x - 4. \quad \ldots \quad \text{(iii)}$$

From (ii), $$2y = 4x + 10,$$

$$y = 2x + 5. \quad \ldots \quad \text{(iv)}$$

Since each is equal to y,

$$\therefore\ 3x - 4 = 2x + 5,$$

$$x = 9.$$

y may now be found from either equation (iii) or (iv).

Thus

$$y = 2x + 5$$

$$= 2 \times 9 + 5.$$

$$= 23.$$

Check this result by substituting these values in equations (i) and (ii).

METHOD II. From one equation obtain the value of one of the unknowns, say y, in terms of the other, and substitute this value in the other equation, thus obtaining an equation with only one unknown.

From (i), $$y = 3x - 4. \quad \ldots \quad \text{(iii)}$$

Substituting in (ii), $$-4x + 2(3x - 4) = 10,$$

$$-4x + 6x - 8 = 10,$$

$$2x = 18,$$

$$x = 9.$$

From (iii), $$y = 3 \times 9 - 4 = 23.$$

METHOD III. In this method the equations are multiplied by such numbers as will make the coefficients of one of the unknowns numerically the same. Then, if the signs of these coefficients

6. A graph of the form $\frac{a}{x} + b$ passes through the points $x = 3$, $y = 3$ and $x = -1, y = -1$. Find its exact equation, and the straight lines which it approaches.

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There are several methods of solving these problems.

are alike, by subtracting, or, if unlike, by adding, this unknown disappears, and an equation is obtained which contains one unknown only.

$$9x - 3y = 12, \quad \ldots \ldots \text{(i)}$$
$$-4x + 2y = 10. \quad \ldots \ldots \text{(ii)}$$

To get rid of y, multiply equation (i) by 2 (the coefficient of y in equation (ii)), and equation (ii) by 3 (the coefficient of y in (i)). Then we have

$$18x - 6y = 24, \quad \ldots \ldots \text{(iii)}$$
$$-12x + 6y = 30. \quad \ldots \ldots \text{(iv)}$$

Adding, $\therefore 6x = 54;$

$$\therefore x = 9.$$

y is obtained from either equation (i) or (ii).

Further Examples.

EXAMPLE i.
$$\frac{x}{3} + 4y = -32,$$
$$6x + 5y = 27.$$

In order to clear the first equation of fractions, multiply both sides of it by 3.

Then
$$x + 12y = -96,$$
$$6x + 5y = 27.$$

These can now be solved by one of the methods given.

EXAMPLE ii.
$$\frac{2}{x} + \frac{5}{y} = 7, \quad \ldots \ldots \text{(i)}$$
$$\frac{3}{x} - \frac{2}{y} = 11. \quad \ldots \ldots \text{(ii)}$$

In such a case it is better to find the value of $\frac{1}{x}$.

Multiply equation (i) by 3 and equation (ii) by 2.

Then
$$\frac{6}{x} + \frac{15}{y} = 21,$$
$$\frac{6}{x} - \frac{4}{y} = 22.$$

$$\text{Subtracting,} \quad \therefore \frac{19}{y} = -1;$$

$$\therefore \frac{1}{y} = \frac{-1}{19},$$

and $$y = -19.$$

$$\text{Substituting in (i),} \quad \frac{2}{x} - \frac{5}{19} = 7,$$

$$\frac{2}{x} = 7 + \frac{5}{19},$$

$$\frac{2}{x} = \frac{138}{19},$$

$$\frac{1}{x} = \frac{69}{19};$$

$$\therefore x = \frac{19}{69}.$$

Verify by substituting these values in equations (i) and (ii).

A modification of this method is that of writing a for $\frac{1}{x}$ and b for $\frac{1}{y}$, and then finding a and b, from which x and y are quickly found by inverting the values found.

EXAMPLE iii.—When there are three unknowns, three different equations are required. From these, unknowns can be eliminated until only one remains. Thus:

$$3a - 2b + c = 1, \quad \ldots \quad \text{(i)}$$

$$2a - 3b - c = -6, \quad \ldots \quad \text{(ii)}$$

$$a + 5b + 3c = 20. \quad \ldots \quad \text{(iii)}$$

Add (i) and (ii), and c is eliminated.

$$\begin{array}{l} 3a - 2b + c = 1 \\ 2a - 3b - c = -6 \\ \hline 5a - 5b \quad = -5 \end{array} \quad \ldots \quad \text{(iv)}$$

Multiply (i) by 3 and subtract (iii).

$$\begin{array}{l} 9a - 6b + 3c = 3 \\ a + 5b + 3c = 20 \\ \hline 8a - 11b \quad = -17 \end{array} \quad \ldots \quad \text{(v)}$$

Equations (iv) and (v) are now readily solved.

To find c substitute the values of a and b in one of the original equations.

The process is the same for any number of unknowns.

Exercise XIV (a)

Solve the following simultaneous equations:

1. $3x - 2y = 18,$
   $2x - 3y = -1.$

2. $3a + 5b = 19,$
   $5a - 4b = 7.$

3. $\frac{x}{3} = \frac{y}{2},$
   $\frac{x}{6} + 2y = 9.$

4. $\frac{a}{5} + 5b = -4,$
   $5a + \frac{b}{5} = 4.$

5. $2x - 5y + 4 = 0,$
   $3x + 2y = 7.$

6. $3x - 2y = 1,$
   $4x + 3y = 41.$

7. $x + \frac{1}{y} = \frac{1}{5},$
   $5x + \frac{2}{y} = \frac{7}{10}.$

8. $\frac{1}{5}(2x + 3y) = \frac{1}{8}(x + 3y + 3) = \frac{1}{2}(9y - x + 1).$

9. $\frac{2x + 1}{7} = \frac{3y - 2}{4},$
   $\frac{x + 2y - 1}{3} = \frac{3x - y + 3}{5}.$

10. $a + 2b - 3c = 6,$
    $2a + 4b - 7c = 9,$
    $3a - b - 5c = 8.$

11. $3(x + y) + 5(x - y) = 19,$
    $5(x + y) - 4(x - y) = 7.$

12. $\frac{a + b}{4} + \frac{a - b}{3} = 10,$
    $\frac{a + b}{8} - \frac{a - b}{6} = 5.$

13. $\frac{2}{3}x - \frac{4}{5}y = 2,$
    $\frac{5}{6}x - \frac{7}{8}y = 4.$

14. $x + 2y = 0,$
    $\frac{8}{x} + \frac{5}{y} = 1.$

15. $7a - 3b = 30,$
    $9b - 5c = 34,$
    $\frac{a}{3} + \frac{b}{5} + \frac{c}{3} = 11.$

16. $\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1,$
    $\frac{5}{x} + \frac{4}{y} + \frac{6}{z} = 24,$
    $\frac{8}{x} - \frac{6}{y} + \frac{6}{z} = 15.$

17. $2x + 4y - 3z = -7,$
$5x + 3y - 2z = 10,$
$7x + 4y - 5z = 3.$

18. $\frac{x}{3} + \frac{y}{4} = \frac{3}{2},$
$2x - y = 19.$

2. Literal Equations.

In Literal Equations, letters other than those which represent the unknowns are introduced. Thus:

EXAMPLE i.—$a(x - b) - b(x - a) = a^2 - b^2$. Find x.

$$ax - ab - bx + ab = a^2 - b^2,$$
$$x(a - b) = a^2 - b^2,$$
$$x = \frac{a^2 - b^2}{a - b} = a + b.$$

In this and similar examples, all the letters, except x, are treated like the numbers in previous examples on equations.

Arrange on one side only all terms containing x. Carry all other terms to the other side. Bracket the terms containing x, and take x outside the bracket. The rest is easy. Verify the above result.

EXAMPLE ii.

$$ax + by = 2ab, \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(i)}$$
$$-bx + ay = a^2 - b^2. \quad . \; . \; . \; . \; . \; . \; . \quad \text{(ii)}$$

To eliminate y, multiply equation (i) by a, and equation (ii) by b, and subtract.

$$a^2x + aby = 2a^2b$$
$$-b^2x + aby = a^2b - b^3$$
$$a^2x + b^2x = a^2b + b^3,$$
$$\therefore \; x(a^2 + b^2) = b(a^2 + b^2);$$
$$\therefore \; x = b.$$

From (i),

$$ab + by = 2ab,$$
$$by = ab;$$
$$\therefore \; y = a.$$

Verify this result by substituting these values in equation (ii).

EXERCISE XIV (B)

1. $a(x - a) - b(x - b) = a - b$. Find x.
2. $\frac{3a^2b}{5cx} = \frac{1}{ab}$. Find x.
3. $b\frac{b - x}{c} - c\frac{c + x}{b} = x$. Find x.

4. $\frac{x}{2a} + \frac{y}{2b} = 1$ and $bx = ay$. Find x and y.

5. $\frac{x}{a} + \frac{y}{b} = \frac{x}{b} + \frac{y}{a}$. Find $\frac{x}{y}$.

6. $x - y = a - b$ and $bx + ay = 2ab$. Find x and y.

7. If $2s = a + b + c$, show that:

(i) $a + b - c = 2(s - c)$.
(ii) $b + c - a = 2(s - a)$.
(iii) $c + a - b = 2(s - b)$.

8. If $c = 2\pi r$, find r.

9. If $A = \frac{\pi d^2}{4}$, find d.

10. If $c = \pi(a + b)$, find a.

11. If $A = \pi ab$, find b.

12. If $A = 4\pi r^2$, find r.

13. If $V = \frac{4}{3}\pi r^3$, find r.

14. $Q = Ws(t_1 - t_2)$.*

Arrange this for calculating respectively:
(i) s. (ii) t_1. (iii) t_2.

15. $Ws(T - x) = w(x - t)$.

Arrange this in convenient forms for calculating:
(i) x. (ii) w. (iii) t. (iv) s.

16. $WL + W(T - x) = w(x - t)$.

Arrange this for the determination of : (i) L. (ii) x.

17. $WL + W(T - x) = w_1(x - t) + w_2 s(x - t)$.

Arrange this for the calculation of : (i) L. (ii) s. (iii) x.

18. $L = l(1 + at)$. Find a.

19. $V = v(1 + bt)$. If $V = \frac{W}{D}$ and $v = \frac{W}{d}$, substitute these values for V and v, and find the equation connecting D and d.

20. $H = k\frac{A(T - t)}{l}$.

Arrange this equation for finding: (i) k. (ii) T.

21. $s = ut + \frac{1}{2}at^2$.

Arrange this equation for calculating : (i) a. (ii) u.
What does the equation become when $u = 0$ and $a = 32$?

* t_1 and t_2 represent two different values of t. The 'subscript' figures 1 and 2 are neither coefficients nor indices.

22. $v^2 - u^2 = 2as$.

Arrange this for determining: (i) s. (ii) a. (iii) v.

23. What does the equation $F = ma$ become when $a = \frac{v-u}{t}$?

Arrange the equation for finding v.

24. $\frac{1}{2}W(v^2 - u^2) = Fs$.

Arrange this for calculating F.

What does the equation become when $F = Wg$?

25. $\frac{W}{v} = d$ and $\frac{W}{V} = D$, and $V = v(1 + ct)$.

Show that $D = \frac{d}{1 + ct}$.

26. $F = \frac{m}{r^2} - \frac{m}{R^2}$. Find m.

27. $\frac{1}{u} + \frac{1}{f} = \frac{1}{v}$.

Arrange this for finding: (i) f. (ii) v.

28. If $\frac{1}{u} - \frac{1}{f} = \frac{1}{v}$, when will v be + and when − ?

29. If $3x - 2 = A(x - 4) + B(x + 1)$ for all values of x, find A and B.

Hint: For finding B, *take* $x = 4$.

30. $y = a + bx$. If, when $y = 113$, x is 230, and when y is 206, x is 320, find a and b.

What is the value of y when $x = 212$?

31. $t = 2\pi\sqrt{\frac{l}{g}}$.

Arrange the equation for finding (i) g. (ii) l.

Find l for $t = 2$ and $g = 32{\cdot}2$.

3. Problems.

Problems, which in many cases appear very difficult when an attempt is made to solve them by the rules of Arithmetic, often yield readily to algebraic treatment.

Consider the following simple example:

When two more passengers enter a railway compartment, it contains three times as many persons as it would have done if, instead, four had alighted. How many were there originally in the compartment.

All such problems contain sufficient information to enable you to represent symbolically all the unknown numbers mentioned, and to form equations from which they can be calculated.

Proceed as follows:

Represent symbolically all the unknown numbers.

It is wise to introduce as few symbols as possible.

Let x = the number of passengers originally present.

Then $(x + 2)$ = the number after 2 more have entered.

and $(x - 4)$ = the number if, instead, 4 had alighted.

Refer to the problem for the relation which exists between these numbers.

The problem states that $(x + 2)$ is three times $(x - 4)$,

$$\text{i.e.}\quad (x + 2) = 3(x - 4),$$
$$x + 2 = 3x - 12,$$
$$-2x = -14,$$
$$x = 7.$$

The result can be verified by testing whether it satisfies the conditions of the problem.

Thus, the problem states that when there are 2 more, i.e. 9 passengers, there are three times as many as there would have been had 4 alighted, i.e. three times 3.

The result satisfies the conditions of the problem.

Statements such as: "One number is so many times another," "One number exceeds another by so much," "The result is the same as," etc., suggest equality, and therefore an equation.

It is important to read the problem carefully, and to write the numbers in symbolic form before attempting to form an equation.

Matters of Importance.

1. State the units when possible.

E.g. Let x = the number of shillings, grams, minutes, etc.

There is no objection to writing £x.

2. The same digit may, in one case, represent units, in another, tens, and so on.

Thus, if the digits of a number be x and y, the number may be $10x + y$, or $10y + x$.

3. As many equations are required as there are symbols representing numbers to be found.

4. Odd and even numbers.

If n represents any number, odd or even, $2n$ will be even, for twice an odd and twice an even number are both even.

It follows that $(2n + 1)$ will always be odd, for the number obtained by adding one to an even number is always odd.

When there are two or more unknowns and the relation between them is not simple enough to allow you to represent them in terms of the symbol chosen for one, it is better to represent them by different symbols. As many equations as there are unknowns are then necessary to determine the unknowns.

EXAMPLE.—*A whole number consists of two digits, and is such that the sum of its digits is one less than one-third of the number, and if 15 is added to twice the number, the digits are reversed. Find the number.*

Let x = the first digit and y = the second digit.
Then, $10x + y$ = the required number.
From the first statement,

$$x + y + 1 = \frac{10x + y}{3}. \quad \ldots \quad \text{(i)}$$

From the second statement,

$$2(10x + y) + 15 = 10y + x. \quad \ldots \quad \text{(ii)}$$

The equations, when simplified, give

$$7x - 2y = 3,$$
$$19x - 8y = -15,$$

from which it will be found that

$$x = 3 \quad \text{and} \quad y = 9.$$

The number is therefore 39.

EXERCISE XIV (c)

1. If a cyclist covers x miles in y hr., how many yards does he cover per minute?
2. If one train travels at the rate of x miles per hour and another at y yd. per minute, what is the difference in their speed in feet per second?
3. If one metre measures 39·37 in., find the difference in yards between x miles and x Km.
4. A gallon of water weighs 10 lb., and a cubic foot, 62·5 lb. Find in gallons the difference between x c. ft. and x gall.
5. When a certain number is increased by 8, the result is the same as when its double is diminished by 1. Find the number.

6. A straight line, 1 ft. long, is divided into two parts such that the difference between the parts is an inch longer than one quarter of the smaller. Find the parts.

7. A certain number consists of two digits, and when 18 is added the digits are reversed. What is the difference between the digits? If the second digit is twice the first, what is the number?

8. If a train had travelled 15 miles an hour faster, it would have journeyed half as far again as it did. Find the speed, and the distance actually covered in 6 hr.

9. The speed of a certain wheel, when running down, is found to decrease proportionally with time. At a certain instant its speed is taken, and again two minutes afterwards, when it is found to have decreased by a quarter. If, in coming to rest, it makes 1200 revolutions from the time the speed was first taken, find the original speed.

10. The sum of four consecutive odd numbers is 48. Find them.

11. Referring to the worked example on p. 167, the digits can be found if it is borne in mind that they are whole numbers. Find them.

12. Find two numbers such that one-third of the first, increased by 6, is equal to one half the second, diminished by 3, and such that their sum is 2 less than five times their difference.

13. If a certain rectangular plot of ground were 4 yd. longer and 2 yd. wider, it would contain 108 more square yards. If it were 6 yd. longer and 6 yd. wider, it would contain 246 more square yards. Find its dimensions.

14. A man buys a dozen eggs, some of which he finds to be bad. Had he received only the good eggs for his money, the price per dozen would have been a third as much again. Find the number of bad eggs.

15. By selling a bicycle, a man gained 5 per cent. What would he have gained per cent had he sold it for half as much again?

CHAPTER XV

FACTORS, FRACTIONS

1. Factors.

It is already known that:

$$(a+b)(c+d) = a(c+d) + b(c+d)$$
$$= ac+ad+bc+bd.$$

Hence the factors of $ac + ad + bc + bd$ are $(a+b)$ and $(c+d)$.

Now commence with the expression $ac + ad + bc + bd$, and retrace the steps.

(i) Bracket in pairs.
(ii) Take the common term a out of the first bracket and b out of the second bracket.
(iii) $c+d$ is common to both terms.

$$(ac+ad)+(bc+bd)$$
$$=a(c+d)+b(c+d)$$
$$=(a+b)(c+d).$$

The factors are $(a+b)$ and $(c+d)$.

Note.—The bracketed expressions in the second line must be exactly alike.

EXAMPLE.—Find the factors of $2ax - ay - 4bx + 2by + 2x - y$.

$$2ax - ay - 4bx + 2by + 2x - y$$
$$=(2ax - ay) - (4bx - 2by) + (2x - y)$$
$$=a(2x - y) - 2b(2x - y) + 1(2x - y)$$
$$=(a - 2b + 1)(2x - y).$$

The factors are $(a - 2b + 1)$ and $(2x - y)$.

EXERCISE XV (A)

Find the factors of:

1. $x^2 - x$. 2. $a - ax$. 3. $a^2 + ax$. 4. $a^2x - ax^2$. 5. $x^3 + x$.

6. $2a^2 - 4ab + 3ac - 6bc$. 7. $ax - ay - 3x + 3y$.

8. $a^4 - a^3 + 3a - 3$. 9. $2x^2y + 2ay - 5x^2 - 5a$.

10. $x^2 - ax - 3a + 3x$. 11. $a^4 + a^3b - 2ab^3 - 2b^4$.

12. $2x^4 + 24x - 8x^2 - 6x^3$. 13. $a^2x + 4bcx - b^2x - 4c^2x$.

14. (i) $a^3 - a^2b + a^2b - ab^2 + ab^2 - b^3$.
(ii) $a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3$.

What, therefore, are the factors of $a^3 - b^3$, and of $a^3 + b^3$?

Factors of a trinomial expression.

EXAMPLE i. $x^2+3x+2.$

If one of these terms can be split into two, then it might be possible to employ the method of grouping.

The middle term $3x$ may be written as $2x + x$; then

$$\begin{aligned} x^2 + 3x + 2 &= x^2 + 2x + x + 2 \\ &= x(x + 2) + 1(x + 2) \\ &= (x + 1)(x + 2). \end{aligned}$$

The difficulty will be in finding how to split up the middle term.

The following rule is sound: *

Multiply the first and last terms of the trinomial; split this product into two factors, whose sum is equal to the middle term.

Thus: (i) $x^2 \times 2 = 2x^2$. (ii) $2x^2 = 2x \times x$, and $2x + x = 3x$.

EXAMPLE ii. $x^2 - 4x - 12.$

First × last $= -12x^2$,

factors $= -6x$ and $+2x$,

sum $= -4x$.

Then,

$$\begin{aligned} x^2 - 4x - 12 &= x^2 - 6x + 2x - 12 \\ &= x(x - 6) + 2(x - 6) \\ &= (x + 2)(x - 6). \end{aligned}$$

EXAMPLE iii. $9x^2 + 3x - 2.$

First × last $= -18x^2$,

factors $= 6x$ and $-3x$,

sum $= 3x$.

$$\begin{aligned} 9x^2 + 3x - 2 &= 9x^2 + 6x - 3x - 2 \\ &= 3x(3x + 2) - 1(3x + 2) \\ &= (3x - 1)(3x + 2). \end{aligned}$$

EXAMPLE iv.—Special case: $a^2 - b^2$.

Here the middle term is missing, i.e. it is 0.

The expression may be written $a^2 \pm 0 - b^2$, or $a^2 \pm 0ab - b^2$.

First × last $= -a^2b^2$,

factors $= ab$ and $-ab$,

sum $= 0$.

$$\begin{aligned} a^2 - b^2 &= a^2 - ab + ab - b^2 \\ &= a(a - b) + b(a - b) \\ &= (a + b)(a - b). \end{aligned}$$

* The rule is based on $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$. Note that $ac \times bd = ad \times bc$.

This last case is so important that it is well to remember it in the following form:

The difference between the squares of two terms is equal to the product of their sum and difference.

Fig. 2 (p. 94) illustrates this result.

Exercise XV (b)

Factorize:

1. $x^2 - 5x + 6$.
2. $x^2 + 5x + 6$.
3. $x^2 + x - 6$.
4. $x^2 + 7x + 6$.
5. $x^2 - x - 6$.
6. $x^2 - 7x + 6$.
7. $x^2 + 5x - 6$.
8. $x^2 - 5x - 6$.
9. $a^2 + 13a + 12$.
10. $a^2 - 13a + 12$.
11. $a^2 + 11a - 12$.
12. $a^2 - 11a - 12$.
13. $a^2 + 8a + 12$.
14. $a^2 - 8a + 12$.
15. $a^2 + 4a - 12$.
16. $a^2 - 4a - 12$.
17. $a^2 + 7a + 12$.
18. $a^2 - 7a + 12$.
19. $a^2 - a - 12$.
20. $a^2 + a - 12$.
21. $x^2 - y^2$.
22. $x^2 - 4y^2$.
23. $4x^2 - y^2$.
24. $x^2 - 1$.
25. $x^2y^2 - 1$.
26. $4a^2 - 9b^2$.
27. $x^2 - (y - 1)^2$.
28. $x^2 - y^2 - 2y - 1$.
29. $9a^2x^2 - 4b^2y^2$.
30. $(a + x)^2 - (a - x)^2$.
31. $(a - x)^2 - (b + y)^2$.
32. $a^2 + 2ax + x^2 - b^2 + 2by - y^2$.
33. $(x^2 + 6x + 9 - y^2 + 4y - 4)$.
34. $x^2 - 4x - y^2 - 6y - 5$.
35. $4x^2 + 12x + 5 - 9y^2 + 12y$.
36. $x^4 + x^2y^2 + y^4$. Observe that

$$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2.$$

37. $x^4 - 3x^2y^2 + y^4$.
38. $x^2 - 2xy - 8y^2$.
39. $6x^2 - 5xy - 6y^2$.
40. $6x^2 + 9xy - 6y^2$.
41. $12(x^2 - y^2) - 7xy$.
42. $(b - c)^3 + 9(c - b)$.
43. $4x^2 + 6xy - 4y^2$.
44. $a^2 - b^2 + ac - bc$.
45. $a(b + c)^2 + b(c + a)^2 + c(a + b)^2 - 4abc$.
46. $(a + x)^4 - (a - 2x)^4$.
47. Show that:

$$4a^2b^2 - (a^2 + b^2 - c^2)^2 = \{(a + b)^2 - c^2\}\{c^2 - (a - b)^2\}$$
$$= (a + b + c)(b + c - a)(c + a - b)(a + b - c).$$

If $a + b + c = 2s$, show that the given expression equals

$$16s(s - a)(s - b)(s - c).$$

48. Starting with $x^2 - (x^2 - 1) = 1$, show that:

$$x + \sqrt{x^2 - 1} = \frac{1}{x - \sqrt{x^2 - 1}}.$$

2. An Important Matter concerning Factors.

Consider the example:

$$x^2 - 4x - 12 = (x - 6)(x + 2).$$

If we substitute 6 for x, then $x - 6$ becomes 0, and therefore $(x - 6)(x + 2)$ becomes 0.

It follows that the expression $x^2 - 4x - 12$ should have a value 0 for x equal to 6.

Verify this statement.

Similarly, the expression $x^2 - 4x - 12$ is equal to 0 when x is equal to -2.

The converse is true, namely: If the value of an expression containing x becomes zero when a value, say a, is substituted for x, then $x - a$ is a factor of the expression.

This gives you another method of testing the accuracy of factors.

EXAMPLES.

(i) To show that $a - b$ is a factor of $a^3 - b^3$.

Substitute b for a; then $a^3 - b^3$ becomes $b^3 - b^3$, which is equal to 0.

(ii) To show that $a - b$ is a factor of

$$a^2(b - c) + b^2(c - a) + c^2(a - b).$$

Substituting b for a, the expression becomes

$$b^2(b - c) + b^2(c - b) + c^2(b - b),$$

which is seen to equal 0.

Similarly, show that $(b - c)$ and $(c - a)$ are factors.

3. Cyclic Order.

In some expressions, the symbols recur in an order called cyclic.

Fig. I shows the symbols a, b, c, spaced round a closed curve.

The symbols of such expressions as those given below follow round the curve in the same direction, namely, clockwise.

(i) $(a - b) + (b - c) + (c - a)$.

(ii) $a(b - c) + b(c - a) + c(a - b)$.

(iii) $ab(a - b) + bc(b - c) + ca(c - a)$.

The sum of the terms of such expressions is often written in a form such as, $\sum_{abc} a(b - c)$, in which Σ (Greek letter sigma) means the sum of terms of the type indicated by the term

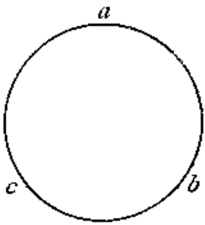


Fig. 1

$a(b - c)$, but completed in cyclic order for the three symbols a, b, c.

Thus, $\sum_{abc} a(b - c)$ is a way of writing briefly Example ii given above.

Exercise XV (c)

1. Show that $2x - 1$ is a factor of $6x^4 + x^3 - 8x^2 + 23x - 10$.
2. Show that $6x^4 + x^3 - 8x^2 + 23x - 10$ is exactly divisible by $x + 2$.
3. Find the factors of $x^4 + 4x^3 - 7x^2 - 22x + 24$.
4. Find the factors of $a^4 - 8a^3 + 17a^2 + 2a - 24$.
5. Show that $(a + b + c)$ is a factor of $a^3 + b^3 + c^3 - 3abc$.
6. Show that $(a - b)$, $(b - c)$, $(c - a)$ are factors of
$$a^3(b - c) + b^3(c - a) + c^3(a - b).$$
Are there other algebraic factors? Why?
7. Write in full the expressions:
$$\sum_{abc} ab(a - b),\ \sum_{abc} a^2(b - c),\ \sum_{abcd} (a - b),\ \sum_{abcd} ab(b - c),\ \sum_{abc} \frac{a}{b - c}.$$
8. Show that $\sum_{abc} (a - b) = 0$.
9. Show that (i) $\sum_{abc} a^2(b - c) = -(a - b)(b - c)(c - a)$.
(ii) $\sum_{abc} ab(a - b) = -(a - b)(b - c)(c - a)$.

4. Application of Factors.

I. Solve the equation $x^2 - 3x + 2 = 0$, i.e. find the values of x for which $x^2 - 3x + 2$ is 0.

By factors, $(x - 2)(x - 1) = 0$.

For a product to give 0 as the result, at least one of the factors must be 0.

Hence $(x - 2)(x - 1)$ equals 0,

(i) when $(x - 2) = 0$, from which $x = 2$;

(ii) ,, $(x - 1) = 0$, ,, $x = 1$.

Check these results by substituting these values of x in turn in $x^2 - 3x + 2$.

Exercise XV (d)

Solve the following equations:

1. $x^2 - 5x + 6 = 0$.
2. $x^2 + 5x = -6$.
3. $2x^2 - 7x + 6 = 0$.
4. $x^2 - x = 12$.
5. $x^2 = 12 + 4x$.
6. $14x = 15 - x^2$.
7. $\frac{x^2}{4} - 3x + 8 = 0$.
8. $3x^2 + 14x = 5$.
9. $x^3 - 2x^2 - 9x + 18 = 0$.
10. $x^2 - 4 = 0$.
11. $x^2 + 4x - 10 = 11$.
12. $x^3 + 8 = 0$.
13. $(x-3)(6x^2 + 5x - 6) = 0$.
14. $x^2 - 3ax + 2a^2 = 0$.
15. $x^2 + ax = 2a^2$.
16. $a^2x^2 + 3ax + 2 = 0$.
17. $(x + 1)^2 = a^2$.
18. $12x^2 - 17x + 8 = 2$.
19. $x^2 - (a + b)x + ab = 0$.
20. $x^4 - 2x^3 + 2x^2 - 3x - 2 = 0$.

II. Examine the expression,

$$x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

It will be observed that, regarded from left to right, the powers of x decrease, and the powers of y increase. This arrangement is very convenient.

If the coefficient of any term is 0, the term vanishes.

If the coefficients of all terms between the end terms are 0, the expression given becomes $x^5 + y^5$. . . . . (ii)

Expression (ii) can be obtained from (i) by adding to (i) terms like the intermediate terms, but of opposite sign, thus:

$$x^5 + y^5 = x^5 + x^4y - x^4y - x^3y^2 + x^3y^2 + x^2y^3 - x^2y^3 - xy^4 + xy^4 + y^5.$$

1. It is readily seen that the right-hand side, and therefore $x^5 + y^5$, is divisible by $x + y$.

$$\begin{array}{r|l} & x^4 \quad -x^3y \quad +x^2y^2 \quad -xy^3 \quad +y^4 \text{ (Quot.)} \\ \hline x+y & x^5+x^4y-x^4y-x^3y^2+x^3y^2+x^2y^3-x^2y^3-xy^4+xy^4+y^5 \\ & x^5+x^4y-x^4y-x^3y^2+x^3y^2+x^2y^3-x^2y^3-xy^4+xy^4+y^5 \\ \hline & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

From the appearance of this example, it is evident that:

(i) Any similar expression having its terms to the same odd power is exactly divisible by $x + y$.

E.g. x^9+y^9 is exactly divisible by $x + y$.

Verify this.

(ii) $x^5 - y^5$ is not exactly divisible by $x + y$. What is the remainder in this case? Similarly, any expression of the form $x^{\text{odd power}} - y^{\text{same odd power}}$, is not exactly divisible by $x + y$.

Examine the above quotient, and observe that:

(i) The terms are conveniently arranged in descending and ascending powers, and that no terms are missing.

(ii) The signs are alternately plus and minus.

The quotient of $\dfrac{x^7 + y^7}{x + y}$ can be written down at once as:

$$x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6.$$

Similarly, $\dfrac{x^3 + y^3}{x + y} = x^2 - xy + y^2.$

The factors of $x^3 + y^3$ are therefore $(x + y)$ and $(x^2 - xy + y^2)$.

2. In the same manner, it is shown that:

$x^5 - y^5$ is exactly divisible by $x - y$.

$$x^5-y^5=x^5-x^4y+x^4y-x^3y^2+x^3y^2-x^2y^3+x^2y^3-xy^4+xy^4-y^5.$$

$$\begin{array}{r|l} & x^4 \quad +x^3y \quad +x^2y^2 \quad +xy^3 \quad +y^4 \\ \hline x-y & x^5-x^4y+x^4y-x^3y^2+x^3y^2-x^2y^3+x^2y^3-xy^4+xy^4-y^5 \\ & x^5-x^4y+x^4y-x^3y^2+x^3y^2-x^2y^3+x^2y^3-xy^4+xy^4-y^5 \\ \hline & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

It follows that $x^{\text{odd}} - y^{\text{same odd}}$ is exactly divisible by $x - y$.

Examine the quotient, and observe that:

(i) The terms run in natural order.

(ii) The signs are all plus.

Question: Referring to the above example, is $x^5 + y^5$ exactly divisible by $x - y$? If not, what is the remainder?

The quotient of $\dfrac{x^3 - y^3}{x - y}$ can be written down at once, thus:

$$\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2.$$

The factors of $x^3 - y^3$ are therefore $(x - y)$ and $(x^2 + xy + y^2)$.

Similarly, verify the following exact divisions:

3. $\dfrac{x^6 - y^6}{x + y} = x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5.$

4. $\dfrac{x^6 - y^6}{x - y} = x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5.$

Observe carefully the signs of the quotients.

Note also that $x^6 + y^6$ is not exactly divisible by either $x + y$ or $x - y$.

Generally, $x^{\text{even}} - y^{\text{same even}}$ is exactly divisible by $x \pm y$.

Summary.

The following are exactly divisible:

1. $\dfrac{x^{\text{odd}} + y^{\text{same odd}}}{x + y}$. Simplest example, $\dfrac{x + y}{x + y} = 1.$

2. $\dfrac{x^{\text{odd}} - y^{\text{same odd}}}{x - y}$. Simplest example, $\dfrac{x - y}{x - y} = 1.$

3. $\dfrac{x^{\text{even}} - y^{\text{same even}}}{x + y}$. Simplest example, $\dfrac{x^2 - y^2}{x + y} = x - y.$

4. $\dfrac{x^{\text{even}} - y^{\text{same even}}}{x - y}$. Simplest example, $\dfrac{x^2 - y^2}{x - y} = x + y.$

Signs. When the signs between the terms are minus in both numerator and denominator, the signs of the quotient are all plus. In all other cases the signs are alternately plus and minus.

In testing whether such expressions are divisible by $x \pm y$, try the expression of the lowest odd or even power.

E.g. $x^{10} - y^{10}$ is divisible by $x + y$, because $x^2 - y^2$ is divisible by $x + y$.

$x^7 - y^7$ is not divisible by $x + y$, because $x - y$ is not divisible by $x + y$.

Or the expressions may be tested by the substitution method given on p. 172.

Thus, $x - y$ is a factor of $x^9 - y^9$, because on substituting y for x,

$$x^9 - y^9 = y^9 - y^9 = 0.$$

It is not a factor of $x^9 + y^9$, because on substituting y for x,

$$x^9 + y^9 = y^9 + y^9 = 2y^9, \text{ not } 0.$$

Exercise XV (e)

What are the factors of:

1. $x^5 + y^5$?
2. $x^5 - y^5$?
3. $x^3 - y^3$?
4. $x^6 - y^6$?
5. $x^4 - y^4$?
6. $x^9 + y^9$?
7. $x^9 - y^9$?

Write down the answers to the following:

8. $\dfrac{x^4 - y^4}{x + y}$.
9. $\dfrac{x^4 - y^4}{x - y}$.
10. $(16x^4 - 81y^4) \div (2x + 3y)$.
11. $(8x^3 - 27y^3) \div (2x - 3y)$.
12. $\dfrac{x^9 - y^9}{x^3 - y^3}$.
13. $\dfrac{x^6 - y^6}{x^2 - y^2}$.
14. $\dfrac{x^6 - y^6}{x^3 - y^3}$.
15. $\dfrac{a^5 - 32b^5}{a - 2b}$.
16. Show that $x - 3$ is a factor of $x^3 - 6x^2 + 5x + 12$. Find the remaining factors.
17. State the factors of $R^3 + r^3$, and of $R^3 - r^3$.
18. Write down the answer to:
$$\frac{(a + b)^3 - 64(c - d)^3}{(a + b) - 4(c - d)}.$$
19. Simplify: $$\frac{(a - b)^4 - (b - c)^4}{a - c}.$$
20. Simplify: $$\frac{(a + b)^3 - 8(b - c)^3}{a - b + 2c}.$$

5. Further Application of Factors.

1. *Highest Common Factors and Lowest Common Multiple of Expressions.*

When given expressions can be factorized, factors common to the expressions are readily found.

Example.—Find the H.C.F. of

$$2x^3 + 6x^2 - 20x, \ 2x^2 - 7x + 6, \ x^2 - 7x + 10.$$

Factorizing each expression, we have:

$$2x^3 + 6x^2 - 20x = 2x(x - 2)(x + 5),$$
$$2x^2 - 7x + 6 = (x - 2)(2x - 3),$$
$$x^2 - 7x + 10 = (x - 2)(x - 5).$$

Examining these factors, it is seen that the factor $(x - 2)$ is common to all the expressions. It is, moreover, the highest common factor.

I.e. the H.C.F. is $(x-2)$. The result can be checked by the method given on page 63.

Example.—Find the L.C.M. of
$2x^3 + 6x^2 - 20x$, $2x^2 - 7x + 6$, $x^2 - 7x + 10$, and $x^2 - 4x + 4$.

Factorizing as before:

$$2x^3 + 6x^2 - 20x = 2x(x - 2)(x + 5),$$
$$2x^2 - 7x + 6 = (x - 2)(2x - 3),$$
$$x^2 - 7x + 10 = (x - 2)(x - 5),$$
$$x^2 - 4x + 4 = (x - 2)(x - 2) = (x - 2)^2.$$

Since the L.C.M. must contain each expression, it must necessarily contain the factors of each expression. Thus, it must contain $2x$, $(x + 5)$, $(2x - 3)$, $(x - 5)$, and also $(x - 2)^2$. If $(x - 2)$ to the first power only is included, the result will not contain the whole of the expression $x^2 - 4x + 4$, but only one factor of it.

The L.C.M. is $2x(x + 5)(2x - 3)(x - 5)(x - 2)^2$.
The result can be checked by division, as shown on p. 63.

2. *Fractions.*

Example i.—Simplify:

$$\frac{a}{a - b} - \frac{2ab}{a^2 - b^2} - \frac{b}{a + b}.$$

$$\text{The given expression} = \frac{a(a + b) - 2ab - b(a - b)}{\text{L.C.M. } (a + b)(a - b)}$$
$$= \frac{a^2 + ab - 2ab - ab + b^2}{(a + b)(a - b)}$$
$$= \frac{a^2 - 2ab + b^2}{(a + b)(a - b)}$$
$$= \frac{(a - b)(a - b)}{(a + b)(a - b)}$$
$$= \frac{a - b}{a + b}.$$

EXAMPLE ii.—Simplify: $\dfrac{\dfrac{x^2 + x - 6}{x^2 - 16}}{\dfrac{x^2 - 5x + 6}{x^2 + 6x + 8}}$.

$$\frac{\dfrac{x^2 + x - 6}{x^2 - 16}}{\dfrac{x^2 - 5x + 6}{x^2 + 6x + 8}} = \frac{(x + 3)(x - 2)}{(x + 4)(x - 4)} \times \frac{(x + 4)(x + 2)}{(x - 3)(x - 2)}$$

$$= \frac{(x + 3)(x + 2)}{(x - 4)(x - 3)}$$

$$= \frac{x^2 + 5x + 6}{x^2 - 7x + 12}.$$

EXAMPLE iii.—Solve: $\dfrac{5}{x - a} - \dfrac{3}{x - b} = \dfrac{2}{x}$.

Multiply both sides of the equation by $x(x - a)(x - b)$, the L.C.M. of the denominators:

$$5x(x - b) - 3x(x - a) = 2(x - a)(x - b),$$
$$5x^2 - 5bx - 3x^2 + 3ax = 2x^2 - 2ax - 2bx + 2ab.$$

Arrange all terms in x^2 and in x on one side:

$$5x^2 - 3x^2 - 2x^2 + 3ax + 2ax - 5bx + 2bx = 2ab.$$

Observe that x^2 vanishes:

$$5ax - 3bx = 2ab,$$
$$x(5a - 3b) = 2ab,$$
$$x = \frac{2ab}{5a - 3b}.$$

EXERCISE XV (F)

Find the H.C.F. and L.C.M. of:

1. $x^2 - 3x - 4$ and $x^2 - 5x + 4$.
2. $x^2 + 2x - 3$, $x^3 - x^2 - 12x$.
3. $2x^2 - 4x - 6$ and $4x^2$.
4. $(x + 2)(x^2 - x - 2)$ and $x^3 - x^2 - 4x + 4$.
5. $6a^2 + a - 2$, $3a^2 + 5a + 2$, $3a^3 - a^2 + a + 2$.
6. $a^2 - b^2$, $a^2 - 3ab + 2b^2$, $a^2 - 2ab + b^2$.
7. $a^3 - b^3$, $a^2 + ab + b^2$, $a^3 + b^3$.
8. Show that: $1 - \dfrac{b}{a + b} = \dfrac{a}{a + b}$.

Reduce to the lowest terms:

9. $\dfrac{x^2 - 3x - 4}{x^2 - 5x + 4}$. (Factorize the numerator and denominator.)

10. $\dfrac{4x^2 - 12ax + 9a^2}{8x^3 - 27a^3}$. 11. $\dfrac{24x^4 - 22x^2 + 5}{48x^4 + 16x^2 - 15}$.

12. Simplify: $\dfrac{x^2 - 9x + 20}{x^2 - 6x} \times \dfrac{x^2 - 13x + 42}{x^2 - 5x}$.

13. Divide: $\dfrac{x^2 - 5x + 6}{x^2 - 5x}$ by $\dfrac{x^2 - 3x}{x^2 - 6x + 5}$.

Simplify:

14. $\left(\dfrac{x - b}{a - b} + \dfrac{x - c}{c - a}\right) \div (c - b)$. 15. $\dfrac{1}{x - 2} - \dfrac{1}{x + 2} - \dfrac{6}{x^2 - 4}$.

16. $\dfrac{2a + 3}{a - a^2} + \dfrac{5 - a}{a - 1} - \dfrac{3}{a}$. 17. $\dfrac{1}{x + 2y} + \dfrac{3y}{x^2 + xy - 2y^2} + \dfrac{y}{(x - y)^2}$.

18. $\left(a + 1 + \dfrac{8}{a - 5}\right)\left(a - 1 - \dfrac{8}{a - 3}\right)$.

19. $\dfrac{4x - 7}{x - 1} - \dfrac{8x - 14}{x + 1} - \dfrac{4x - 7}{x^2 - 1}$. 20. $\dfrac{ax}{a - b} - x$.

21. $\dfrac{1}{x - y} + \dfrac{2y}{x^2 - y^2}$. 22. $\dfrac{x - 3}{x^2 + 5x + 6} - \dfrac{x + 3}{x^2 - x - 6}$.

23. (i) $\dfrac{a^3}{(a - b)(b - c)} + \dfrac{b^3}{(b - c)(b - a)} + \dfrac{c^3}{(c - a)(c - b)}$.

(ii) $\sum_{abc} \dfrac{a + b}{(b - c)(c - a)}$.

24. Find: $\dfrac{x^2 - 2x + 1}{x^2 - 5x + 6} \times \dfrac{x^2 - 4x + 4}{x^2 - 4x + 3} \times \dfrac{x^2 - 6x + 9}{x^2 - 3x + 2}$.

25. Divide $\dfrac{a}{a + b} + \dfrac{b}{a - b}$ by $\dfrac{a^2}{a^2 - b^2} - \dfrac{b^2}{a^2 + b^2}$.

26. From $\dfrac{xy}{x^2 + 2xy + y^2}$ take $\dfrac{y}{x + y}$.

27. Simplify: $\dfrac{a^2 - b^2}{2(a + b)} - \dfrac{a^2 + b^2}{2a - 2b} + \dfrac{2a^2b + 2ab^2}{a^2 - b^2}$.

Solve:

28. $12 - \frac{5x - 10}{7x} = \frac{35}{x} - 22\frac{2}{7}$. 29. $\frac{1}{x-1} + \frac{2}{x-3} = \frac{3}{x-2}$.

30. $\frac{ax}{a+b} - x = \frac{b^2x}{a^2 - b^2} - \frac{ab}{a-b}$.

31. Find A and B such that:

$$\frac{12x - 5}{6x^2 - 5x - 6} = \frac{A}{3x+2} + \frac{B}{2x-3}.$$ (*See Ex. XIV* (B), 29.)

CHAPTER XVI

SURDS

1. There are some roots which cannot be determined exactly, and which are therefore most conveniently treated as algebraic numbers.

EXAMPLES.—$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{4}$, etc.

If an attempt is made to extract these roots, it will be found that their decimal portion neither terminates nor recurs. Neither can the roots be expressed in vulgar fraction form. They are said to be **Irrational.**

Such roots are called **Surds.**

EXERCISE XVI (A)

Calculate correct to five decimal places:

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$.

2. Fundamental Examples.

(i) *Addition and subtraction of unlike surds.*

$\sqrt{2}$ added to $\sqrt{5} = \sqrt{5} + \sqrt{2}$.

$\sqrt{2}$ subtracted from $\sqrt{5} = \sqrt{5} - \sqrt{2}$.

The above are similar to adding and subtracting a and b.

Using the numbers obtained in Exercise XVI (A), the pupil should convince himself that $\sqrt{5} + \sqrt{2}$ does not equal $\sqrt{7}$, and that $\sqrt{5} - \sqrt{2}$ does not equal $\sqrt{3}$.

(ii) *Addition and subtraction of like surds.*

$\sqrt{2}$ added to $\sqrt{2} = 2\sqrt{2}$. Compare, $a + a = 2a$.
$3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$. Compare, $3a + 2a = 5a$.
$5\sqrt{2} - 2\sqrt{2} = 3\sqrt{2}$. Compare, $5a - 2a = 3a$.

Remember that in $3\sqrt{2}$, the figure 3 is a coefficient.

(iii) *Products and quotients of unlike surds.*

$$\sqrt{3} \times \sqrt{2} = \sqrt{3 \times 2} = \sqrt{6}, \quad 2\sqrt{3} \times 5\sqrt{2} = 10\sqrt{6},$$

$$\sqrt{6} \div \sqrt{2} = \sqrt{\frac{6}{2}} = \sqrt{3},$$

$$\frac{10\sqrt{6}}{2\sqrt{3}} = \frac{10}{2} \times \frac{\sqrt{6}}{\sqrt{3}} = \frac{10}{2} \times \sqrt{\frac{6}{3}} = 5\sqrt{2}.$$

(iv) *Products and quotients of like surds.*

$$\sqrt{2} \times \sqrt{2} = 2, \qquad \sqrt{3} \times \sqrt{3} = 3,$$
$$3\sqrt{2} \times 2\sqrt{2} = 3 \times 2 \times \sqrt{2} \times \sqrt{2} = 6 \times 2 = 12,$$
$$\frac{6\sqrt{2}}{2\sqrt{2}} = \frac{6}{2} \times \frac{\sqrt{2}}{\sqrt{2}} = 3 \times 1 = 3.$$

(v) *Simplification of surds*

Sometimes a given surd has a factor which is rational. In such a case, the factor can be placed as a coefficient, as shown in the following examples:

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}.$$
$$\sqrt{243} = \sqrt{81 \times 3} = \sqrt{81} \times \sqrt{3} = 9\sqrt{3}.$$

Exercise XVI (b)

1. Verify the foregoing examples by making use of the numbers found in the last exercise.
2. Express in simpler form:
$$\sqrt{27}, \quad \sqrt{48}, \quad \sqrt{125}, \quad \sqrt{252}.$$
3. Given that $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$, $\sqrt{5} = 2{\cdot}236$, find $\sqrt{8}$, $\sqrt{27}$, $\sqrt{48}$, $\sqrt{12}$, $\sqrt{45}$.
4. Simplify: (i) $3\sqrt{2} - 5\sqrt{3} + 6\sqrt{2} + \sqrt{3} - \sqrt{27}$.
(ii) $\sqrt{7} + 3\sqrt{3} - \sqrt{245} + \sqrt{12} - \sqrt{63} + \sqrt{45} + \sqrt{28}$.

5. (i) $10\sqrt{3} \times 2\sqrt{3} =$? (ii) $\sqrt{2} \times \sqrt{3} \times \sqrt{5} =$?
(iii) $(\sqrt{3} + \sqrt{2}) \times \sqrt{3} =$? (iv) $\frac{2\sqrt{3}}{3\sqrt{2}} \times 6\sqrt{6} =$?

6. $(2\sqrt{3} + 5\sqrt{2})(3\sqrt{3} - 2\sqrt{2}) =$?
Compare $(2a + 5b)(3a - 2b)$.

7. Expand $(\sqrt{a} + \sqrt{b})^2$ and $(\sqrt{a} - \sqrt{b})^2$.

8. (i) Add $3\sqrt{5}$ to $3\sqrt{2}$. (ii) From $8\sqrt{3}$ take $3\sqrt{3}$.

9. To $6\sqrt{5}$ add $\sqrt{125}$.

10. (i) Add together $\sqrt{3} - 1$ and $\sqrt{3} + 1$.
(ii) Find the difference between $\sqrt{3} - 1$ and $\sqrt{3 - 1}$.

11. Evaluate: $3(\sqrt{3} + \sqrt{2}) - \sqrt{2}(2 - \sqrt{2}) + \sqrt{3}(3 - \sqrt{3})$.

12. (i) Find the difference between $10\sqrt{\cdot 2}$ and $\sqrt{2}$, $\sqrt{2}$ and $10\sqrt{\cdot 02}$.
(ii) Determine in decimal form:

$$\sqrt{\cdot 02},\quad \sqrt{\cdot 2},\quad \sqrt{\cdot 03} + \sqrt{\cdot 02},\quad \left(\frac{\sqrt{\cdot 03}}{\sqrt{\cdot 02}} - \sqrt{\cdot 03} \times \sqrt{\cdot 02}\right).$$

13. Solve the equation, $3\sqrt{x} + 2\sqrt{3} = \sqrt{2x} - \sqrt{2}$.

14. What are the factors of $x^2 - 3$ and of $2x^2 - 3$?

3. Applications.

To evaluate $\frac{5}{\sqrt{3}}$.

It is advisable to avoid surds as divisors.

In order to remove $\sqrt{3}$ from the denominator, multiply both numerator and denominator by $\sqrt{3}$. Thus:

$$\frac{5}{\sqrt{3}} = \frac{5 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{5\sqrt{3}}{3} = \frac{5 \times 1{\cdot}732}{3}$$
$$= 2{\cdot}8866\ldots = 2{\cdot}887 \text{ (approx.)}.$$

This process is known as **Rationalization** of the denominator.

4. Conjugate Surds.

The following example is very important:

$$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2$$
$$= 3 - 2$$
$$= 1.$$

Compare this with the general example:

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b.$$

Observe that no surd appears in the product.

The sum of and the difference between two surds are said to be conjugate.

The use of conjugate surds is shown in the following example.

Evaluate: $$\frac{5\sqrt{3}+3\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}.$$

Multiply the numerator and the denominator by the conjugate of the denominator. Then:

$$\frac{5\sqrt{3}+3\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}=\frac{5\sqrt{3}+3\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}\times\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}=\frac{30+21\sqrt{6}+18}{(3\sqrt{2})^2-(2\sqrt{3})^2}$$

$$=\frac{48+21\sqrt{6}}{18-12}$$

$$=\frac{48+21\sqrt{6}}{6}$$

$$=8+\tfrac{7}{2}\sqrt{6}.$$

Observe that the final result is much simpler than the original expression, and that to obtain the answer no division by a decimal is necessary.

Complete the computation.

EXERCISE XVI (c)

Calculate in the shortest manner:

1. $\frac{2}{\sqrt{5}}$, $\frac{1}{\sqrt{3}}$, $\frac{6}{\sqrt{7}}$, $\sqrt{\frac{2}{3}}$. 2. $\frac{1}{\sqrt{2}-1}$, $\frac{1}{1+\sqrt{2}}$, $\frac{\sqrt{2}}{1+\sqrt{2}}$.

3. $\frac{\sqrt{2}}{3-\sqrt{2}}$, $\frac{1+\sqrt{2}}{\sqrt{2}-1}$.

4. Find the value of $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b},$$ when $a=\sqrt{3}$, $b=\sqrt{3}-1$, $c=\sqrt{3}+1$.

5. Simplify: $\frac{\sqrt{2}}{2}-\frac{1}{\sqrt{2}}$.

6. Multiply: (i) $(2\sqrt{3}+3\sqrt{2})$ by $(2\sqrt{3}-3\sqrt{2})$.
   (ii) $(a\sqrt{3}+b\sqrt{2})$ by $(a\sqrt{3}-b\sqrt{2})$.
   (iii) $\left(\frac{2}{\sqrt{3}}+\frac{3}{\sqrt{2}}\right)$ by $\left(\frac{2}{\sqrt{3}}-\frac{3}{\sqrt{2}}\right)$.

7. What is the value of $x^2-\frac{1}{x^2}$, when $x=2+\sqrt{3}$?

8. Rationalize the denominators, and evaluate the following:

(i) $\frac{2+\sqrt{2}}{2-\sqrt{2}}$. (ii) $\frac{2\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

(iii) $\frac{2\sqrt{5}+\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$. (iv) $\frac{2\sqrt{3}+3\sqrt{2}}{3\sqrt{5}-2\sqrt{3}}$.

9. Find the value of $(2\sqrt{2}+\sqrt{3})(\sqrt{6}-4)$.

10. Simplify: $\frac{7+\sqrt{21}}{7-\sqrt{21}}+\frac{\sqrt{21}-3}{\sqrt{21}+3}$. 11. Calculate: $\frac{(\sqrt{3}-\sqrt{2})^2}{2-\sqrt{2}}$.

12. Find the value of $\frac{x+y}{x-y} \div \frac{x-y}{x+y}$,

when $x = \sqrt{3+\sqrt{2}}$ and $y = \sqrt{3-\sqrt{2}}$.

5. If the side of a square is a and the diagonal d, then

$$d = a\sqrt{2}.$$

If it is required to find the side in terms of the diagonal, then $a = \frac{d}{\sqrt{2}}$, which, when rationalized, becomes

$$a = \frac{d\sqrt{2}}{2}.$$

6. If one of the angles of an equilateral triangle is bisected, and the bisector produced to meet the opposite side, then each triangle formed has angles 60°, 30° and 90° respectively. In fig. 1, let BC be of unit length. Then:

AB is 2 units and AC is $\sqrt{2^2-1^2} = \sqrt{3}$ units.

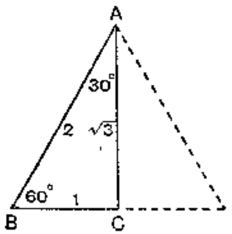


Fig. 1

It follows that all triangles similar to this have their sides in the ratio of $1:2:\sqrt{3}$, or, in order of size, $1:\sqrt{3}:2$.

EXAMPLE i.—In a given 60°, 30°, right-angled triangle, the shortest side is 10 cm. Find the remaining sides.

$$\frac{AB}{BC} = \frac{2}{1},$$
$$AB = 2BC;$$
$$\therefore AB = 20 \text{ cm.}$$

$$\frac{AC}{BC} = \frac{\sqrt{3}}{1},$$
$$AC = BC\sqrt{3};$$
$$\therefore AC = 10\sqrt{3} \text{ cm.}$$

EXAMPLE ii.—In a given 60°, 30°, right-angled triangle, the side opposite the angle 60° is 10 in. Find the remaining sides.

$$\frac{BC}{AC} = \frac{1}{\sqrt{3}},$$
$$BC = \frac{AC}{\sqrt{3}};$$
$$\therefore BC = \frac{10}{\sqrt{3}},$$
$$\text{i.e. } BC = \frac{10\sqrt{3}}{3} \text{ in.}$$

$$\frac{AB}{BC} = \frac{2}{1},$$
$$AB = 2BC;$$
$$\therefore AB = \frac{20\sqrt{3}}{3} \text{ in.}$$

EXERCISE XVI (D)

1. Find the ratio of the sides of a right-angled isosceles triangle.
2. By means of a right-angled triangle, find a straight line equal to $\sqrt{a^2 + \frac{a^2}{4}}$, when a is the length of a given straight line.
3. The hypotenuse of a 60°, 30°, right-angled triangle is 10 cm. Find the remaining sides.
4. The diagonal of a square courtyard measures 40 yd. Find its area and the length of its sides.
5. Find in surd form the trigonometrical ratios of 30°, 45° and 60°.
6. In fig. 2, find CD and the area of the triangle CDB, in terms of x.
7. In fig. 3, find CD and the shaded area, in terms of $AB = x$.
8. The slant edges of a square pyramid make angles of 60° with the base. Calculate their length if the side of the base is 5 in. long.

9. The usual way of fitting a circular filter paper is to fold it into quadrants and then to open it in the form of a cone. Show that the apex angle of a vertical section of this cone through the apex is 60°.

If the diameter of the paper is 5 in., find the altitude of the cone.

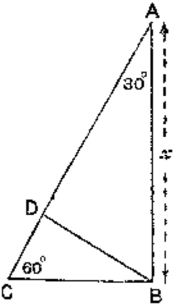


Fig. 2

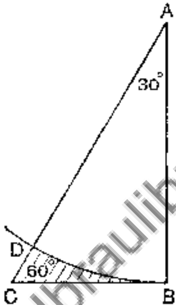


Fig. 3

CHAPTER XVII

LOGARITHMS, THE SLIDE RULE

1. Logarithms.*

It has been mentioned on p. 49, that logarithm is another name for index. More particularly, the logarithm of a number is the index of the power to which another number, called the base, must be raised to equal the given number.

When 10 is chosen as the base, the logarithms are called common logarithms.

Number	·01	·1	1	10	100	1000	10000
log base 10	−2	−1	0	1	2	3	4

* Logarithms were invented by John Napier, a Scotsman (1550–1617 A.D.). Their use was extended by Henry Briggs (1615).

Examining the above table, starting from the right and working towards the left, you notice that as the numbers are divided by 10, the logarithms decrease by 1. It follows, naturally,

that the log of 1 to the base 10 is 0, i.e. $\log_{10}1 = 0$.
" " ·1 " " −1, " $\log_{10}\cdot1 = -1$,
" " ·01 " " −2, " $\log_{10}\cdot01 = -2$.

Let us prove this:

$1 = \frac{10}{10}$, i.e. $\frac{10^1}{10^1} = 10^{1-1} = 10^0$, since we subtract indices when dividing powers of the same base;

$$\therefore \log_{10} 1 = 0.*$$

Similarly, $$\cdot1 = \frac{1}{10} = \frac{10^0}{10^1} = 10^{0-1} = 10^{-1};$$

$$\therefore \log_{10}\cdot1 = -1.$$

EXAMPLE.—In the same way, show that $\log_{10}\cdot01 = -2$, and $\log_{10}\cdot0001 = -4$.

It follows from the table that

(i) The logarithms of numbers greater than 1 are positive.
(ii) The logarithms of numbers less than 1 are negative.

2. Now consider numbers between these powers of 10.

Take the number 35, it lies between 10 and 100. Its logarithm is between 1 and 2, i.e. it is 1 plus a decimal.

Similarly, 350 lies between 100 and 1000, and its logarithm is between 2 and 3, and ·08 lies between ·01 and ·1, and therefore its logarithm is between −2 and −1.

QUESTIONS

Between what numbers are the logarithms of

2675, 3, ·6, ·06, ·0006?

3. A graph of numbers and their logarithms can be constructed from the following numbers which should be checked by the student.

$$\sqrt{10} = 10^{\cdot5} = 3\cdot162$$
$$10^{\cdot25} = \sqrt{3\cdot162} = 1\cdot778$$
$$10^{\cdot125} = \sqrt{1\cdot778} = 1\cdot334$$

* Similarly, the log of 1 to any base is 0, for $b^0 = 1$.

$$10^{\cdot 75} = 10^{\cdot 5} \times 10^{\cdot 25} = 3{\cdot}162 \times 1{\cdot}778 = 5{\cdot}623$$
$$10^{\cdot 875} = 10^{\cdot 75} \times 10^{\cdot 125} = 5{\cdot}623 \times 1{\cdot}334 = 7{\cdot}499.$$

No.	10	7·5	5·62	3·16	1·78	1·33
log	1	·875	·75	·5	·25	·125

From the graph (fig. 1) verify that log 2 is ·3, log 5 is ·7 and log 6·3 is ·8, approximately.

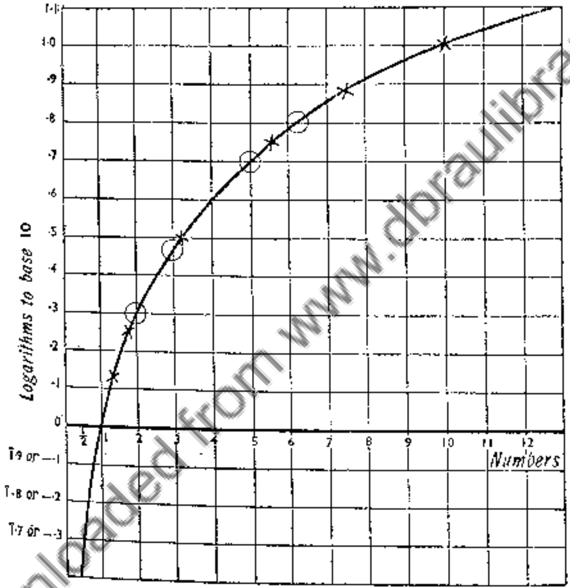


Fig. 1

4. With these numbers we can illustrate several important uses of logarithms.

(i) To find log 20.

$$\begin{aligned} 20 &= 10 \times 2 \\ &= 10^1 \times 10^{0\cdot 3} \quad (\log_{10}2 = \cdot 3, \text{ i.e. } 10^{0\cdot 3} = 2) \\ &= 10^{1\cdot 3}, \quad \text{adding indices} \end{aligned}$$

i.e. $\log_{10}20 = 1{\cdot}3$.

(ii) To find log 63.

$$\begin{aligned} 63 &= 10 \times 6{\cdot}3 \\ &= 10^{1} \times 10^{\cdot 8} \\ &= 10^{1\cdot 8}, \end{aligned}$$

$$\text{i.e. } \log_{10} 63 = 1{\cdot}8.$$

(iii) To find log 63000.

$$\begin{aligned} 63000 &= 6{\cdot}3 \times 10000 \\ &= 10^{\cdot 8} \times 10^{4} \\ &= 10^{4\cdot 8}, \end{aligned}$$

$$\therefore \log_{10} 63000 = 4{\cdot}8.$$

5. You will notice that the decimal part of the logarithm, viz. ·8, is the same for 6·3, 63, 630, 6300, 63000, etc., i.e. *it does not change with the change of the position of the decimal point in the number.*

6. The whole number part of the logarithm, however, does depend upon the position of the decimal point. Thus, for 6·3 it is 0; for 63, 1; for 630, 2; 6300, 3; etc.

7. The name given to the decimal portion of a logarithm is **mantissa**, and that given to the whole number, **characteristic.**

Hence we see that the **characteristic** depends upon the position of the decimal point. Thus, in the numbers given, it is one less than the number of figures before the decimal point.

Number	Number of figures before the decimal point	Characteristic
6·3	1	0
63·	2	1
630·	3	2
6300·	4	3
63000·	5	4
	etc.	

EXERCISE (*Oral*)

1. log 2 = ·3 approx. State the logs of 20, 200, 2000, 20000, 200000, 2000000.
2. log 5 = ·7 approx. State the logs of 50, 500, 5000, 50000, 500000, 5000000.

8. Let us see for what other numbers we can find the logs from those of 2, 5 and 6·3.

(i) $4 = 2 \times 2$
$= 10^{0·3} \times 10^{0·3}$
$= 10^{0·6}$,
i.e. $\log 4 = 0·6$.

(ii) $12·6 = 2 \times 6·3$
$= 10^{0·3} \times 10^{·8}$
$= 10^{1·1}$,
i.e. $\log 12·6 = 1·1$.
Notice that the characteristic is correct.

(iii) $2·5 = \frac{5}{2}$
$= \frac{10^{·7}}{10^{·3}}$
$= 10^{·4}$,
i.e. $\log 2·5 = ·4$.

(iv) $1·26 = \frac{6·3}{5}$
$= \frac{10^{·8}}{10^{·7}}$ Subtracting indices (division)
$= 10^{·1}$,
i.e. $\log 1·26 = ·1$.
Compare this mantissa with that of log 12·6.

Exercise XVII (a)

1. Find log 25, log $(5)^3$, i.e. log 125, log$(2)^4$. Can you state the rule for finding the log of a power of a number from the log of the number?
2. Write down log 40, log 4000 and log $\frac{1·26}{2}$.
3. Find the log of (5×2), and of $(6·3 \times 5)$. Can you write down the rule for finding the log of the product of given numbers from the logs of the numbers?
4. Write down log 1·26, 126, 1260.
5. Find the log of $\frac{6·3}{2}$, $\frac{63}{5}$, $\frac{10}{6·3}$, $\frac{250}{63}$, $\frac{625}{8}$.

   Can you write down the rule for finding the log of a quotient from the logs of the numbers?
6. Write down log 25, 250, 25000.

By referring to the graph, verify that the mantissa of each of your results is approximately correct.

9. Roots as Powers.

We have seen on p. 48 that $\sqrt{a} \times \sqrt{a} = a$ or a^1.
Now, if we represent $\sqrt{a}$ as a to some power, the index denoting

the power will be such that the sum of two such indices is 1. It follows that each is $\frac{1}{2}$. In other words, $\sqrt{a}$ equals $a^{\frac{1}{2}}$.

Another method of obtaining the same result is as follows:

Let $\sqrt{a} = a^x$; then $\sqrt{a} \times \sqrt{a} = a^1$,

i.e. $a^x \times a^x = a^1$.

But $a^x \times a^x = a^{(x+x)}$, or a^{2x};

$\therefore\ 2x = 1$ and $x = \frac{1}{2}$.

Similarly, $\sqrt[3]{a} = a^{\frac{1}{3}}$, since $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a^1$;

and $\sqrt{a^3} = a^{\frac{3}{2}}$, since $a^{\frac{3}{2}} \times a^{\frac{3}{2}} = a^{\frac{3}{2}+\frac{3}{2}} = a^3$.

Application to Logarithms.

(i) To find $\log_{10}\sqrt{2}$.

From the graph, $\log_{10}2 = \cdot 3$,

i.e. $2 = 10^{\cdot 3}$;

$\therefore\ \sqrt{2}$ or $2^{\frac{1}{2}} = \sqrt{10^{\cdot 3}} = 10^{\frac{\cdot 3}{2}} = 10^{\cdot 15}$,

i.e. $\log_{10}\sqrt{2} = \cdot 15$.

Now $\sqrt{2} = 1\cdot 414$.

Hence $\log_{10}1\cdot 414 = \cdot 15$.

It will be observed that $\log_{10}\sqrt{2} = \frac{1}{2}\log_{10}2$.

(ii) To find $\log_{10}\sqrt[3]{5}$.

$$\sqrt[3]{5} = \sqrt[3]{10^{\cdot 7}} \quad (\text{since } \log_{10}5 = \cdot 7)$$
$$= 10^{\frac{\cdot 7}{3}}.$$

Hence $\log_{10}\sqrt[3]{5} = \frac{\cdot 7}{3}$ i.e. $\frac{1}{3}\log_{10}5$.

10. Summary.

(i) The logarithm of a product is obtained by adding the logarithms of the factors,

i.e. $\log(xy) = \log x + \log y$.

(ii) The logarithm of a quotient is obtained by subtracting the logarithm of the divisor from that of the dividend. Thus:

$$\log\frac{x}{y} = \log x - \log y.$$

er of a number is obtained by
he number by the index of the

$g\, x,$

$(x^{\frac{1}{n}}) = \frac{1}{n}\log x.$

ed as a fractional power.

XVII (B)

oot signs:

$\sqrt[5]{a}, \quad \sqrt{(a+b)^3}.$

$- b^{\frac{1}{2}}).$

$\log_{10}\sqrt[3]{2}.$

$\sqrt[3]{2^2}, \quad \log_{10}\sqrt[3]{5^2}.$

rmine:

$_{10}15, \quad \log_{10}8, \quad \log_{10}9,$

$_{10}\frac{10}{2}, \quad \log_{10}\frac{10}{3}, \quad \log_{10}33\frac{1}{3}.$

e Decimal.

$\frac{2}{10};$

$\log 2 - \log 10$

$\cdot 3 - 1.$

r a reason to be explained later, it
orm, but as $\bar{1}\cdot 3$.
r the characteristic, and the man-

$\log_{10}0\cdot 2$ is the same as that of $\log_{10}2$.

$\frac{5}{1000};$

$= \log 5 - \log 1000$

$= 0\cdot 7 - 3$

$= \bar{3}\cdot 7.$

is $\bar{1}\cdot 7$, $\log 0\cdot 05$ is $\bar{2}\cdot 7$, etc.
eristic of the logarithm of a pure

decimal (i.e. a number less than unity) is negative, and is numerically one more than the number of noughts immediately after the decimal point. Thus:

Number	Number of noughts directly after the decimal point	Characteristic of log
·5	0	$\bar{1}$
·05	1	$\bar{2}$
·005	2	$\bar{3}$
·0005	3	$\bar{4}$
	etc.	

Remember that the **mantissa** of a logarithm is positive.

12. Operations involving Negative Characteristics.

(i) To find the logarithm of $(0{\cdot}2 \times 6{\cdot}3)$.

$$\log(0{\cdot}2 \times 6{\cdot}3) = \log 0{\cdot}2 + \log 6{\cdot}3$$
$$= \bar{1}{\cdot}3 + 0{\cdot}8$$
$$= \begin{array}{r} \bar{1}{\cdot}3 \\ +0{\cdot}8 \\ \hline 0{\cdot}1 \end{array}$$

Adding the mantissae, we get 1·1, which is all positive. Then $\bar{1} + 1 = 0$.

Remember that numbers carried forward from the mantissae are positive.

(ii) To find $\log \dfrac{6{\cdot}3}{0{\cdot}2}$.

$$\log \frac{6{\cdot}3}{0{\cdot}2} = \log 6{\cdot}3 - \log 0{\cdot}2$$
$$= \begin{array}{r} 0{\cdot}8 \\ -\bar{1}{\cdot}3 \\ \hline 1{\cdot}5 \end{array}$$

Subtracting the mantissae, we get ·5. Subtracting $\bar{1}$ from 0, we get +1.

(iii) To find $\log \dfrac{0{\cdot}05}{0{\cdot}063}$.

$$\log \frac{0{\cdot}05}{0{\cdot}063} = \log 0{\cdot}05 - \log 0{\cdot}063$$
$$= \begin{array}{r} \bar{2}{\cdot}7 \\ -\bar{2}{\cdot}8 \\ \hline \bar{1}{\cdot}9 \end{array}$$

Subtracting ·8 from 1·7, we get ·9. Pay back 1 to $\bar{2}$ and we get $\bar{1}$ to subtract from $\bar{2}$, which gives $\bar{1}$.

The operation is actually:

$$\begin{array}{rr} & \bar{2} + 1{\cdot}7 \\ \text{Subtract,} & (\bar{2}+1) + \ {\cdot}8 \\ \hline & \bar{1} + \ {\cdot}9 \end{array}$$

carried out from the equivalent form:

$$\begin{array}{r} \bar{3} + 1{\cdot}7 \\ -(\bar{2} + {\cdot}8) \\ \hline \bar{1} + {\cdot}9 \end{array}$$

log 0·05 × $\bar{2}$·7 ·4

{Multiply ·7 by 2; result, + 1·4. Multiply $\bar{2}$ by 2; result, $\bar{4}$. $\bar{4}$ + 1·4 = $\bar{3}$·4.

$\frac{1}{3}$ log 0·02 = $\frac{1}{3}$ × $\bar{2}$·3.

cteristic is not exactly divisible, the ently carried out by changing the visible characteristic, in this case $\bar{3}$, positive number, in this case 1, to

$\frac{1}{3}(\bar{3} + 1{\cdot}3) = \bar{1}{\cdot}43.$

$\frac{1}{7}(\overline{21} + 5{\cdot}6) = \bar{3}{\cdot}8.$

CISE XVII (c)

ristics of the logarithms of the fol-53, 0·00063, 0·00505, 0·5, 0·006003.

there immediately after the decimal logarithms of which have the follow-

$\bar{2}$, $\bar{6}$, 3, $\bar{8}$?

4, $\bar{1}$·064 and 1·952,

3, $\bar{3}$·826 and 2·473.

From $\bar{1}$·28 subtract 0·76.

From $\bar{1}$·125 subtract $\bar{2}$·941.

and $\bar{2}$·634 by 10.

1 by 7, and $\bar{1}$·61 by 3.

13. Logarithm Tables.

So far we have used convenient numbers only, and their logarithms to the first place of decimals.

It is explained in a later chapter how logarithms may be calculated. Tables of logarithms are given at the end of this book, but the student will find it convenient and time-saving to have a separate book of tables at his elbow.

The following is an extract from the table of logarithms:

LOGARITHMS

First 2 Figures	Third Figure										Fourth Figure								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	9	13	17	21	26	30	34	38
..	..	..	..	..	..	..	..	..	..	..	..				..	..			
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3	5	8	10	13	15	18	20	23

The tables do not give the characteristic of a logarithm, but the **mantissa** only.

Look at line 17. From this line, the mantissae of the logarithms of numbers of four digits, the first two of which are 1 and 7, are obtained. Thus:

(i) For log 17, we take the number next to 17, viz. 2304.

At the head of the column in which this number is found, is the figure 0. You will remember that the mantissa of log 17 is the same as for log 170 and for log 0·17, etc.

The mantissa, being a decimal, is really ·2304.

The characteristic you must determine for yourself.

E.g. $\log 17 = 1{\cdot}2304$, $\log 1{\cdot}7 = 0{\cdot}2304$, $\log 1700 = 3{\cdot}2304$, $\log 0{\cdot}0017 = \bar{3}{\cdot}2304$, $\log 0{\cdot}17 = \bar{1}{\cdot}2304$, etc.

(ii) If we require log 173, then we take the number on the 17 line, which is in the column headed 3. The number is 2380.

Then $\log 173 = 2{\cdot}2380$, $\log 1{\cdot}73 = 0{\cdot}2380$,
$\log 0{\cdot}0173 = \bar{2}{\cdot}2380$, etc.

(iii) If the number has a fourth figure, and is, say, 1738, then having got the log of 173, we move along the 17 line until we reach the next set of numbers having at the head of the columns the figures 1 to 9. These numbers have to be added to those in the other columns. Thus, in the 17 line, in the column headed 8, the number 20 is found. Adding this number to 2380, the mantissa of log 1738 is obtained, namely 2400.

(iv) For single numbers such as 5, 7, etc., use the lines 50, 70, etc., since the mantissae of the logarithms of these tens are the same as for the unit figures.

(v) If the number has more than four significant figures the nearest fourth is used. E.g. 173862 is regarded as 173900 and the log. as 5·2403.

Constant practice in the use of these tables is essential.

Exercise XVII (d)

Using the tables, determine the logs of the following numbers:

1. 1·65, 16·5, 165, 1650, 0·165, 0·0165, 0·00165.
2. 3, 30, 300, 0·3, 328, 32·8, 0·00328.
3. 89·61, 8961, 0·8961, 5060, 5·006, π.
4. 1, 0·1, 0·001, 10·01, 100·2, 1002.

14. Antilogarithms.

There are also tables under the title "Antilogarithms". These give the numbers corresponding to given logarithms and are used in exactly the same way as the logarithm tables. Remember that only the mantissa of the log is used in determining the digits which build up the number. Thus:

(i) To find the antilog of 1·2475, that is to find the number of which the log is 1·2475.

In the antilog tables, find the number corresponding to the mantissa ·2475, viz. 1766 + 2, i.e. 1768. These are the digits composing the number, but the position of the decimal point depends upon the characteristic. In this case the characteristic is 1, and this we know to be one less than the number of figures before the decimal point. There are, therefore, 2 figures before the decimal point, and the number of which the log is 1·2475 is therefore 17·68.

(ii) To find antilog $\bar{3}$·0456.

Referring to the tables, the antilog of ·045 is 1109. For the next figure 6, add 2, and the antilog of ·0456 is found to be 1111.

The characteristic $\bar{3}$, being negative, shows that the number is a decimal fraction, and that there are two noughts directly after the decimal point.

Hence, antilog $\bar{3}$·0456 is 0·001111.

It is in the placing of the decimal point that most mistakes are made. Before leaving your answer, check it by seeing that the characteristic of its log agrees with that given.

Exercise XVII (e)

From the tables, find the antilogs of:

1. 0·1684, 2·1684, $\bar{2}$·1684, $\bar{1}$·1684.
2. 1·5672, $\bar{1}$·5672, 3·5672.
3. Arrange −0·7313 so that only the characteristic is negative, then determine its antilog.
4. Of what number is 0 the logarithm?
5. Arrange −2·3642 so that only the characteristic is negative, then determine its antilog.
6. Determine by logs $(2{\cdot}63)^{-3}$.
7. Using logarithms, calculate $\frac{1}{6{\cdot}034}$ and $\frac{1}{38{\cdot}35}$.
8. Calculate $(326{\cdot}4)^{-\frac{1}{3}}$.

15. Applications.

Remember that logarithms cannot be applied to sums and differences.

Example i.—Evaluate $\frac{83{\cdot}69 \times 2{\cdot}685 \times 0{\cdot}384}{97{\cdot}64 \times 0{\cdot}067}$.

The following is a convenient way to set out the computation. Write A for the answer, then

$$\log A = \overset{\text{Numerator}}{(\log 83{\cdot}69 + \log 2{\cdot}685 + \log 0{\cdot}384)}$$

$$\overset{\text{Denominator}}{-(\log 97{\cdot}64 + \log 0{\cdot}067)}$$

$$= \begin{Bmatrix} 1{\cdot}9227 \\ 0{\cdot}4289 \\ \bar{1}{\cdot}5843 \end{Bmatrix} \qquad - \begin{Bmatrix} 1{\cdot}9896 \\ \bar{2}{\cdot}8261 \end{Bmatrix}$$

$$\begin{array}{rr} 1{\cdot}9359 & \quad 0{\cdot}8157 \\ -0{\cdot}8157 & \\ = \ 1{\cdot}1202 & \end{array}$$

$$\therefore\ A = \text{antilog } 1{\cdot}1202$$
$$= 13{\cdot}19.$$

EXAMPLE ii.—Evaluate $\sqrt[3]{\dfrac{83{\cdot}69 \times 0{\cdot}2685 \times 0{\cdot}0384}{97{\cdot}64 \times 0{\cdot}67}}$.

$$\log A = \tfrac{1}{3}[(\log 83{\cdot}69 + \log 0{\cdot}2685 + \log 0{\cdot}0384) - (\log 97{\cdot}64 + \log 0{\cdot}67)]$$

$$= \frac{1}{3}\left[\begin{array}{lcl} \left\{\begin{array}{r} 1{\cdot}9227 \\ \bar{1}{\cdot}4289 \\ \bar{2}{\cdot}5843 \end{array}\right. & - & \left\{\begin{array}{r} 1{\cdot}9896 \\ \bar{1}{\cdot}8261 \end{array}\right. \\ \hline \quad \bar{1}{\cdot}9359 & & \quad 1{\cdot}8157 \\ -1{\cdot}8157 & & \\ \hline \quad \bar{2}{\cdot}1202 & & \end{array}\right]$$

$$= \frac{\bar{3} + 1{\cdot}1202}{3}$$

$$= \bar{1}{\cdot}3734;$$

$$\therefore \; A = \text{antilog}\, \bar{1}{\cdot}3734$$

$$= 0{\cdot}2362.$$

EXAMPLE iii.—Evaluate

$$\sqrt{\frac{6{\cdot}728 \times (2{\cdot}87)^3}{15{\cdot}34}} + \frac{1{\cdot}374 \times \sqrt{46{\cdot}47}}{729{\cdot}23}.$$

The two parts connected by the plus sign must be evaluated separately and the results added. Let the answers to the parts be denoted by A_1 and A_2 respectively.

$$\log A_1 = \tfrac{1}{2}[\log 6{\cdot}728 + 3 \log 2{\cdot}87 - \log 15{\cdot}34]$$

$$= \frac{1}{2}\left[\begin{array}{l} 0{\cdot}8279 + 3 \times 0{\cdot}4579 - 1{\cdot}1858 \\ + 1{\cdot}3737 \\ \hline \quad 2{\cdot}2016 \\ - 1{\cdot}1858 \\ \hline \quad 1{\cdot}0158 \end{array}\right]$$

$$= 0{\cdot}5079;$$

$$\therefore \; A_1 = \text{antilog}\, 0{\cdot}5079$$

$$= 3{\cdot}221.$$

$$\log A_2 = \log 1{\cdot}374 + \tfrac{1}{2}\log 46{\cdot}47 - \log 729{\cdot}2$$

$$= \left\{\begin{array}{r} 0{\cdot}1379 + \dfrac{1{\cdot}6672}{2} - 2{\cdot}8628 \\ +\,0{\cdot}8336 \\ \hline 0{\cdot}9715 \\ -\,2{\cdot}8628 \end{array}\right.$$

$$= \bar{2}{\cdot}1087$$

$$\therefore\ A_2 = \text{antilog}\,\bar{2}{\cdot}1087$$

$$= 0{\cdot}01284,$$

$$A_1 + A_2 = \begin{array}{r} 3{\cdot}221 \\ +0{\cdot}01284 \\ \hline 3{\cdot}2338 \end{array}$$

Exercise XVII (f)

Using logarithms, compute:

1. $\dfrac{3{\cdot}28 \times 15{\cdot}23}{8{\cdot}42}$.   2. $\dfrac{\pi(6{\cdot}25)^2}{2{\cdot}85}$.   3. $\dfrac{263 \times 10{\cdot}8 \times 496}{4{\cdot}198\pi}$.

4. $\sqrt[3]{3{\cdot}082}$.   5. $15(6{\cdot}32)^{1{\cdot}34}$.   6. $\pi\sqrt{\dfrac{126{\cdot}3 \times 2{\cdot}005}{3{\cdot}28 \times 54{\cdot}8}}$.

7. $(6{\cdot}345 \times 0{\cdot}1075)^{2{\cdot}5} \div (0{\cdot}00374 \times 96{\cdot}37)^3$.

8. (i) $\sqrt[3]{20760}$.   (ii) $\sqrt[3]{0{\cdot}02076}$.   9. $\dfrac{3{\cdot}024 \times \sqrt{0{\cdot}1275} \times 73{\cdot}24}{\sqrt[3]{2{\cdot}124} \times 32{\cdot}78}$.

10. Solve the equation $5^x = 120$.

11. (i) $(1{\cdot}03)^{50}$.   (ii) $(\frac{2}{19})^{\frac{1}{5}}$.   (iii) $(1{\cdot}04)^7$.

12. $(22{\cdot}15 \div 4{\cdot}139)^{0{\cdot}86}$.   13. $(55{\cdot}21)^2 \times 3{\cdot}142 \div 2{\cdot}206$.

14. $\left(\dfrac{28{\cdot}68 \times 0{\cdot}0173}{0{\cdot}00197}\right)^{0{\cdot}4}$.   15. $\sqrt[5]{\dfrac{\tan 40^\circ}{65}}$.

16. Find x when $2^{x+1} = 3^{x-1}$.

17. $\dfrac{3{\cdot}862 \times \sqrt{13{\cdot}25}}{11{\cdot}28} + 3{\cdot}52$.   18. $\dfrac{\pi(3{\cdot}85)^2}{0{\cdot}58} - \dfrac{\sqrt{6{\cdot}54}}{0{\cdot}76}$.

19. $12{\cdot}68\sqrt{0{\cdot}057} + \log\tan 55^\circ$.

20. $\log\dfrac{\sin 60^\circ}{\cos 45^\circ} + \log(\sin 60^\circ \cos 45^\circ)$.

Work also Exercise I(c), 3 to 12, using logarithms.

16. A Simple Slide Rule.

Take two strips of cardboard, and on an edge of each mark off lengths corresponding to the logarithms of the numbers from 1 to 10, and then of the tens from 10 to 100.

This can be done from the graph on p. 189 by placing the strips along the ordinates, and marking off their lengths.

The lengths for the tens can be obtained at once from those of the units; for, say, $\log 50 = \log 10 + \log 5$, i.e. the length for 50 is the sum of the lengths for 10 and 5.

The scale constructed is called a logarithmic scale. The lengths upon it represent the logarithms of the numbers, and are not proportional to the actual numbers.

See that the scales are marked as shown in the figure.

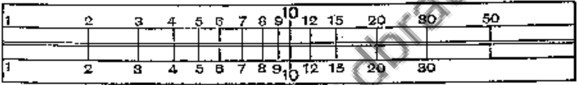


Fig. 2

To check the rule, find the products of simple numbers, say 2 and 3, in the following manner:

Move the lower scale to the right until its mark 1 is opposite 2 on the upper scale. Then the reading on the upper scale, which is opposite the 3 on the lower scale, should be the product of 2 and 3. Notice that in this operation the logarithms of 2 and 3 have been added.

Division is the reverse operation. To divide 6 by 3, place the lower scale so that the 3 mark is opposite the 6 mark on the upper scale. Then the mark on the upper scale to which the 1 on the lower scale is opposite is the quotient.

Now mark on the scales, lengths corresponding to 12, 15, 16, 18, 25, 35, 45, etc.

In order to test your rule, find by its means:

(i) 5×3, (ii) 5^2, (iii) 8×5, (iv) $\cdot 5 \times 3$ $\left(\text{i.e. } \frac{5 \times 3}{10}\right)$.

There are several good makes of slide rules on the market that give results sufficiently accurate for practical purposes. Books of instruction are supplied with the rules.

CHAPTER XVIII

THE QUADRATIC GRAPH

1. Graphs of Expressions containing the second, but no higher power.

Take the simplest expression, x^2, and plot its values for different values of x (fig. 1).

-4	-3	-2	-1	$=x=$	0	1	2	3	4
16	9	4	1	$=x^2=$	0	1	4	9	16

Examine the graph, and verify the following statements:

(i) The graph is not a straight line, but a curve with a vertex and two diverging branches.

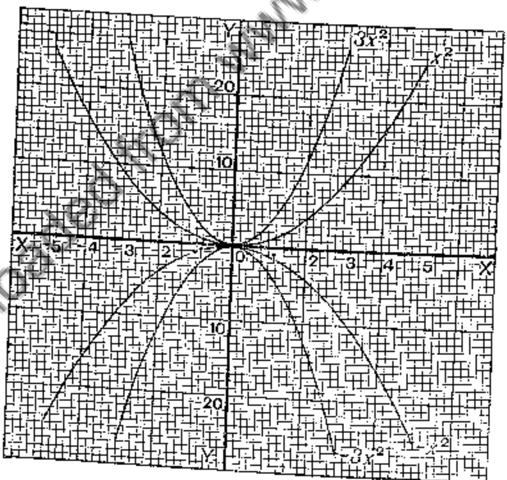


Fig. 1

(ii) The values of x^2 are all positive.

(iii) The graph touches the axis of x, the vertex, in this case, being at the origin.

(iv) The graph is symmetrical about the axis of y; i.e. if the paper is folded so that the crease is along the axis of y, one branch of the curve will coincide with the other branch.

(v) The curve grows straighter away from the vertex.

(vi) The gradient changes.

(vii) The gradient is positive (up) on the right, zero at the vertex, and negative (down) on the left.

2. On the same axes, plot the graph of $3x^2$, and compare it with that of x^2.

Observe: (i) The gradient of this graph changes more rapidly.

(ii) The coefficient 3 appears as the value of $3x^2$ when x equals 1 or -1.

3. On the same axes, plot the graphs of $-x^2$ and $-3x^2$, and compare them with those of x^2 and $3x^2$.

You will conclude that the inversion is due to the change in sign.

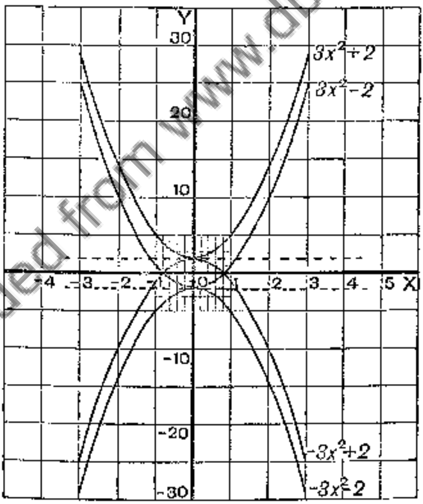


Fig. 2

4. To determine the effect of an added constant, plot the graphs (fig. 2) of:

(i) $y = 3x^2 + 2$. (ii) $y = 3x^2 - 2$.

(iii) $y = -3x^2 + 2$. (iv) $y = -3x^2 - 2$.

Comparing these with the other graphs, it is seen that:

(i) The effect of the 2 has been to raise the graph $3x^2$ through a distance 2 above the origin.

Observe that the graph has the same shape, and is still symmetrical about the axis of y, also that the minimum value of $3x^2 + 2$ is 2, this being shown by the position of the vertex.

If a new axis of x is drawn through the point $y = 2$, then the coefficient 3 appears as the new value of y when x is 1.

(ii) The effect of the -2 has been to lower the graph of $3x^2$ through a distance 2 below the origin.

The graph of $3x^2 - 2$ cuts the axis of x at two points, and at these the value of $3x^2 - 2$ is 0. The intersections of the axis of x give the corresponding values of x. Observe that one is positive and the other negative.

These values of x are called the roots of the equation,

$$3x^2 - 2 = 0,$$

from which

$$3x^2 = 2,$$

$$x^2 = \tfrac{2}{3},$$

and

$$x = +\sqrt{\tfrac{2}{3}} \text{ or } -\sqrt{\tfrac{2}{3}}.$$

For compactness, the last line is usually written in the form

$$x = \pm\sqrt{\tfrac{2}{3}}.$$

The sign $\pm$ is read *plus or minus*.

The graph of $3x^2 + 2$ shows that the equation $3x^2 + 2 = 0$ has no roots, because the graph does not intersect the axis of x. In such cases the roots are said to be imaginary.

(iii) The graph of $-3x^2 + 2$ shows that there are two roots to the equation

$$-3x^2 + 2 = 0.$$

(iv) The graph of $-3x^2 - 2$ shows that the roots of the equation

$$-3x^2 - 2 = 0$$

are imaginary.

Summary.

1. The graph of an expression of the form $ax^2 \pm c$ is a curve. It is called a **Parabola.**

2. The graph is symmetrical about the axis of y.

3. If the coefficient of x^2 is positive, the vertex is downwards; if negative, upwards.

4. When the added constant is positive, the vertex is above the origin; when negative, below the origin.

5. To straighten the graph of $ax^2 + c$.

If the values of $3x^2$ are plotted against the values of x^2 instead of x, the graph shown in fig. 3 is obtained.

-3	-2	-1	$=x=$	0	1	2	3	4
9	4	1	$=x^2=$	0	1	4	9	16
27	12	3	$=3x^2=$	0	3	12	27	48

The graph is seen to be a straight line of gradient 3.

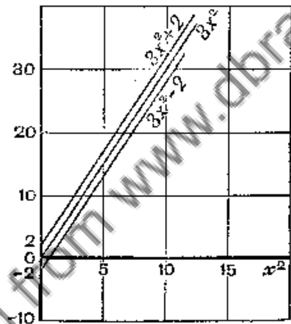


Fig. 3

Similarly, plot the values of

(i) $3x^2 + 2$, (ii) $3x^2 - 2$, (iii) $-3x^2 + 2$, (iv) $-3x^2 - 2$, against x^2.

Examine the graphs, and verify that:

(i) All the graphs are straight lines.
(ii) The gradient in each case is the coefficient of x^2.
(iii) The added constant is shown on the axis of y.

That the graph should be a straight line will be readily understood if, say, z is substituted for x^2.

Then, $3x^2 = 3z$, and the equation of the graph becomes $y = 3z$.

This result is very useful in testing whether a given set of numbers follow an assumed law.

Suppose that when certain values depending upon x are plotted against x, a graph is obtained which looks like those of expressions containing x^2; then, in order to test your supposition, it is only necessary to plot the values against x^2 instead of x.

If the resulting graph is a straight line, your assumption is correct.

Moreover, the expression can be written down at once, for the gradient gives the coefficient of x^2, and the intersection with the axis of y the added constant.

Application.

A good example is found in establishing the law of the pendulum.

The following numbers, showing the time of swing of simple pendulums of different lengths, were obtained by a class of boys aged 13 to 14 years:

Time (sec.), t	1·05	1·6	1·76	2	2·45
Length (cm.), l	25	50	75	100	150

On plotting l against t, the graph obtained appeared to be one of the branches of a parabola. When l was plotted against t^2 the graph was found to be approximately a straight line, the gradient of which was 25 and the added constant 0. Verify this.

The relation is therefore $l = 25t^2$, or $t = 0{\cdot}2\sqrt{l}$.

This agrees closely with the formula

$$t = 2\pi\sqrt{\frac{l}{g}}, \text{ where } g = 981.$$

6. In order to determine the effect of introducing the first power of x into the expression, plot the graph of x^2+x (fig. 4).

−3	−2	−1	$=x=$	0	1	2	3
6	2	0	$=(x^2+x)=$	0	2	6	12

Comparing this graph with that of x^2, it is seen that the effect of x is to displace, as it were, the graph of x^2 towards the second quadrant, and to lower it.

The graph cuts the axis of x at two points, and the axis of symmetry is no longer the axis of y, but an axis mid-way between the two values of x for which $x^2 + x = 0$, i.e. at $x = -\frac{1}{2}$.

Observe also that for this value of x the value of $x^2 + x$ is a minimum.

The axis of symmetry is readily found as follows:

$$x^2 + x = \{x^2 + x + (\tfrac{1}{2})^2\} - (\tfrac{1}{2})^2 = (x + \tfrac{1}{2})^2 - \tfrac{1}{4}.$$

If z is written for $(x + \frac{1}{2})$, the expression becomes $z^2 - \frac{1}{4}$. If z is measured along the axis of x, but from the point $x = -\frac{1}{2}$,

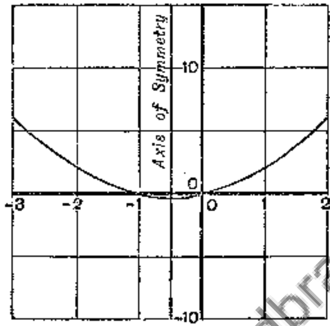


Fig. 4

and if the expression $z^2 - \frac{1}{4}$ is measured on a vertical axis through this point, the graph of $z^2 - \frac{1}{4}$ is symmetrical about this vertical axis, and cuts it at the point $-\frac{1}{4}$.

Thus the axis of symmetry passes through the point on Ox for which $x = -\frac{1}{2}$.

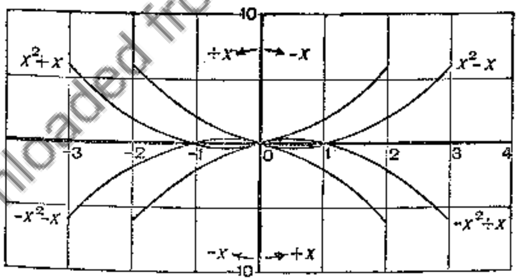


Fig. 5

Plot and examine the graphs of:

(i) $x^2 - x$. (ii) $-x^2 + x$. (iii) $-x^2 - x$.

Verify that the vertices of the graphs of x^2 and $-x^2$ are displaced in the counter-clockwise direction by $+x$, and in the clockwise direction by $-x$ (fig. 5).

7. We are now able to examine the graph of an expression containing x^2, x and a constant.

Plot the graph of, say, $2x^2 - 4x - 3$:

-2	-1	$=x=$	0	1	2	3	4
13	3	$=(2x^2-4x-3)=$	-3	-5	-3	3	13

The axis of symmetry evidently passes mid-way between $x = 0$ and $x = 2$, i.e. through $x = 1$.

Draw the axis, make it a new axis of y, and reckon values along the axis of x from this new axis of y.

Call the new values z; then

$$z = (x - 1) \quad \text{and} \quad x = (z + 1).$$

Substituting this value of x in the given expression, we have

$$\begin{aligned} 2x^2 - 4x - 3 &= 2(z + 1)^2 - 4(z + 1) - 3 \\ &= 2z^2 - 5. \end{aligned}$$

This expression contains only the second power of z.

Notice that -5 is the minimum value of the original expression.

If, therefore, values of $2z^2 - 5$, i.e. of $2x^2 - 4x - 3$, are plotted against z^2, the graph is a straight line of gradient 2, which intersects the new axis of y at -5 (fig. 6).

The original equation can be obtained from the equation to this straight line by substituting $(x - 1)$ for z.

The importance of this is, that we can test whether the equation of a given graph is of the form $y = ax^2 + bx + c$, a, b and c being constants.

The method is as follows:

(i) Draw the axis of symmetry.
(ii) Reckon values of x from the axis of symmetry.
(iii) Plot the values of y against the squares of the new values of x.

If the graph (iii) is a straight line, the assumption is correct. From the equation to graph (iii) the equation of the original graph can be found.

8. A quadratic expression, i.e. an expression of the form ax^2+bx+c, can be determined if its values for three known values of x are known.

Examples.—The values of a quadratic expression are 6, 2 and 3, when the values of x are 1, -1 and -2 respectively. Find the expression. Substitute the known values in the general equation:

$$ax^2 + bx + c = y;$$

then
$$a + b + c = 6, \text{ when } x = 1.$$
$$a - b + c = 2, \text{ when } x = -1.$$
$$4a - 2b + c = 3, \text{ when } x = -2.$$

Solve these simultaneous equations for a, b and c.
It will be found that a is 1, b is 2, and c is 3.
The expression is therefore $x^2 + 2x + 3$.

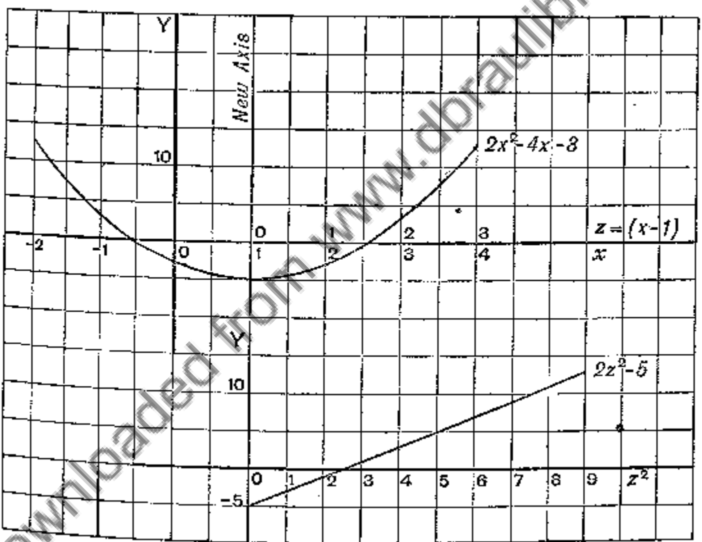


Fig. 6

Exercise XVIII

Find the quadratic expressions which have the given values for the stated values of x:

1. -5, when x is 1; 40, when x is -4; -2, when x is 2.
2. -3, when x is 0: 6, when x is 3; -19, when x is -2.

Find functions of x which have the following values for the given values of x:

3. 15, when x is -2; 10, when x is 3; 1, when x is 0.

4. -5, when x is 1; -5, when x is -1; -6, when x is 0.

5. Draw a graph of $x = y^2$.

6. Express $x^2 + 2x + 3$ as a function of z, where $z = (x + 1)$.

7. Plot the graph of $2x^2$, and on the same axes, the graph of $5x - 3$.

Add the ordinates of the two graphs, and compare the graph obtained with that of $2x^2 + 5x - 3$.

8. The following are corresponding values of x and y. Find the equation connecting them:

x	-3	-2	-1	0	1	2	3
y	6	3	2	3	6	11	18

9. Trace the graphs of $y = 2x^2$ and $y = 5x + 3$, and from them find the roots of the equation $2x^2 - 5x = 3$.

10. Draw a graph of $y = x(a - x)$ from $x = 0$ to a. From the graph, show that if the sum of two positive numbers is given, their product is increased by making the numbers more nearly equal.

11. The following numbers show the weight of circular sheet iron discs of different diameters. Plot the numbers, and find the law connecting weight and diameter.

Diameter (cm.)	1	2	4	6	8	10
Weight (gm.)	·785	3·14	12·57	28·27	50·28	78·55

12. The distance through which a body falls under gravity is given by the following table:

Time (sec.)	0	1	2	3	4	5
Distance (ft.)	0	16	64	144	256	400

Measure distance on the vertical axis, and time on the horizontal axis, and construct a graph showing the relation between distance and time.

The velocity acquired by the body is given in the following table:

Time (sec.)	0	1	2	3	4	5
Velocity (ft. per. sec.)	0	32	64	96	128	160

Represent the velocity on the other side of the axis used for distance in the distance-time graph, and using the same axis of time, plot on the same figure the graph of velocity and time.

By means of these graphs, find the velocity acquired by the body when it has fallen through (i) 36 ft., (ii) 100 ft., (iii) 200 ft., (iv) 320 ft.; and the distance through which it has fallen when its velocity is (i) 40 ft. per sec., (ii) 80 ft. per sec., (iii) 100 ft. per sec., (iv) 136 ft. per sec.

13. Show by Algebra, in the manner used for $x^2 + x$, that on the axis of symmetry of the graph $x^2 - 4x - 3$ the value of x is 2, and that the minimum value is therefore -7.

14. Express $2x^2 + 4x - 3$ in the form $2z^2 \pm$ a number. What is the value of z? Then show that the axis of symmetry is the line $x = -1$, and that the minimum value of $2x^2 + 4x - 3$ is therefore -5.

15. The value of the bending moment (M) at a distance x from the centre of a beam of length l, supported at each end and loaded uniformly at w lb. per foot, is given by the equation

$$M = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right).$$

What do you know concerning the graph obtained by plotting M and x?

For what value of x will M be a maximum?

Choosing suitable values for w and l, construct the graph.

16. When a beam fixed at one end only is uniformly loaded at w lb. per unit length, the bending moment (M) at a distance x from the free end is given by the equation

$$M = -\frac{wx^2}{2}.$$

Construct a graph connecting M and x, and draw as many conclusions from it as you can.

17. Why is the axis of symmetry of the graph of $ax^2 + bx + c$ the same as that of the graph of $ax^2 + bx$?

18. Draw the graphs of $y = x^2 - 4$ and $y = 4 - x^2$. What do the points of intersection indicate?

CHAPTER XIX

QUADRATIC EQUATIONS

A quadratic equation contains the second but no higher power of the unknown. It may or may not contain the first power.

1. Pure Quadratic.

A pure quadratic equation contains only the second power of the unknown.

EXAMPLE. $3x^2 = 108.$

To solve this equation, find x^2, and then extract the square root.

Thus:

$$\begin{aligned} 3x^2 &= 108, \\ x^2 &= 36, \\ x &= \pm\sqrt{36}. \\ &= \pm 6. \end{aligned}$$

Substituting these values for x, it will be found that both $+6$ and -6 satisfy the equation.

2. Adfected Quadratic.

An adfected quadratic equation contains both the second and the first power of the unknown.

EXAMPLE. $x^2 + 2x - 8 = 0$ or $x^2 + 2x = 8$.

It has been seen already that such equations can be solved graphically, or by the method of factors (p. 174).

There is another method, which is more useful when the factors are not at once apparent.

It consists of arranging so that the side of the equation containing the unknown is a perfect square; then, on taking the square root, the equation is reduced to one containing the first power of the unknown only.

EXAMPLE.—Solve the equation $2x^2-7x-22=0$.

(i) Take the number -22 to the other side, leaving on the left the terms containing x.	$2x^2 - 7x = 22,$
(ii) Reduce the coefficient of x^2 to 1, by dividing both sides by 2.	$x^2 - \frac{7}{2}x = 11,$
(iii) Complete the square on the left, by adding the third term, viz. $(-\frac{7}{4})^2$. Add the same number to the other side.	$x^2 - \frac{7}{2}x + (-\frac{7}{4})^2 = 11 + \frac{49}{16},$ $(x-\frac{7}{4})^2 = \frac{225}{16},$
(iv) Extract the square root of both sides.	$(x-\frac{7}{4}) = \pm\sqrt{\frac{225}{16}} = \pm\frac{15}{4},$ $x = \frac{7}{4} \pm \frac{15}{4},$

i.e. $x = \frac{7}{4} + \frac{15}{4} = 5\frac{1}{2}$ or $x = \frac{7}{4} - \frac{15}{4} = -2$.

Substituting these values of x in the given equation, it will be found that the equation is satisfied by either value of x.

EXERCISE XIX (A)

Solve the equations. (*Check your answers by substitution.*)

1. $(x-1)^2 = 4$.
2. $3(x+1)^2 = 27$.
3. $(x+1)^2 = 1$.
4. $(x-3)^2 = 0$.
5. $2x^2 - 16x + 32 = 0$. What do you notice concerning the roots? Account for it if you can.
6. $x^2 + 21x + 120 = 10$.
7. $6x^2 + 13x = -6$.
8. $6x^2 - 1 = 5x$.
9. $x(x-2) + 4x = 3$.
10. $6x^2 - 17x + 12 = 0$.
11. $10x^2 + 11x - 35 = 0$.
12. $4x^2 + 3x - 5 = 2$.
13. $(4x-1)(2x+3) - (10x+3)(5-x) = 2(3x-7)^2 + 25 - 94x^2$.

In 14 to 24, clear away the fractions first.

14. $\frac{1}{x+2} + \frac{2}{2x+1} + \frac{4}{2x-1} = 0$.
15. $\frac{x}{2} + \frac{x-1}{3} + \frac{x-2}{4x} = 1$.
16. $\frac{1}{x-2} - \frac{1}{x+4} = \frac{1}{x}$.
17. $\frac{x}{x+3} + \frac{x+3}{x+5} = 0$.
18. $\frac{x+3}{x-3} - \frac{x-3}{x+3} = 6\frac{6}{7}$.
19. $\frac{3x}{x+1} = 6 - \frac{4x+1}{2x+1}$.
20. $\frac{7x}{x+2} = \frac{5x+1}{3x+1}$.
21. $\frac{2-3x}{2+3x} - \frac{2+3x}{2-3x} = 2$.

22. $\dfrac{3x+5}{3x-1}-\dfrac{5x^2}{9x^2-1}=\dfrac{3x+8}{3x+1}$. 23. $\dfrac{1}{2(x-1)}+\dfrac{3}{x^2-1}=\dfrac{1}{4}$.

24. $\dfrac{5x-7}{10x-5}=\dfrac{1}{10}-\dfrac{4x-3}{4x-2}$. 25. $\dfrac{2}{x^2}+\dfrac{5}{x}=18$.

3. The General Quadratic Equation.

In the general quadratic equation, the coefficients and the term not containing x are represented by letters, say a, b and c. The general equation is, then,

$$ax^2+bx+c=0.$$

This can be solved by the method of completing the square, thus:

$$ax^2+bx=-c,$$

$$x^2+\frac{bx}{a}=-\frac{c}{a}.$$

Complete the square by adding $\left(\dfrac{b}{2a}\right)^2$ to each side:

$$x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a},$$

$$\text{i.e.}\quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2};$$

$$\therefore\ x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}.$$

That is, $x=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ or $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

If this result be remembered, the roots of quadratic equations can be written down immediately.

Thus: Write down the roots of $12x^2-5x-2=0$.

Here $a=12$, $b=-5$ and $c=-2$.

$$\therefore\ x=\frac{-(-5)+\sqrt{(-5)^2-4\times 12\times -2}}{2\times 12}$$

$$\text{or}\quad \frac{-(-5)-\sqrt{(-5)^2-4\times 12\times -2}}{2\times 12}$$

$$=\frac{5+\sqrt{25+96}}{24}\quad\text{or}\quad\frac{5-\sqrt{25+96}}{24}$$

$$=\frac{2}{3}\quad\text{or}\quad -\frac{1}{4}.$$

Return to Exercise XV (D), and write down the roots from those of the general equation, by the method just given.

The Discriminant.

The roots of the general quadratic equation $ax^2 + bx + c = 0$ are, as we have just seen:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The part $b^2 - 4ac$ is so important that it is called the **Discriminant.** The nature of the roots depends upon the discriminant.

For example:

(1) If $b^2 - 4ac$ (the discriminant) $= 0$, the roots of the equation are *equal*, each being equal to $-\frac{b}{2a}$.

(2) If $b^2 - 4ac$ is negative, the roots are said to be *imaginary* since there is no real square root of a negative number. On the other hand, if $b^2 - 4ac$ is positive, the roots are said to be *real*.

These results and others are illustrated graphically in the next chapter.

4. Applications.

1. A chord at right angles to the diameter of a circle.

Such a chord is bisected by the diameter.

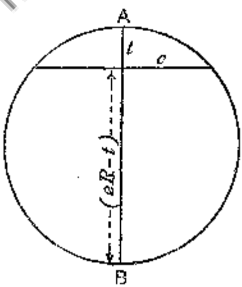


Fig. 1

Let the radius of the circle (fig. 1) be R, the length of the semi-chord, c, and the shorter portion of the diameter cut off by the chord, t.

The other portion of the diameter is, then, $(2R - t)$.

By an important theorem in Geometry (p. 124),

$$c^2 = (2R - t)t,$$

from which

$$c = \pm\sqrt{(2R - t)t},$$

The whole chord is, therefore, $\pm 2\sqrt{(2R - t)t}$.

EXAMPLE.—In a circle of 5 cm. radius, a chord of length 6 cm. is drawn at right angles to a diameter. Find where the chord intersects the diameter.

From

$$c^2 = (2R - t)t,$$
$$t^2 - 2Rt = -c^2.$$

Solving this quadratic,

$$t^2 - 2Rt + (-R)^2 = R^2 - c^2,$$
$$t - R = \pm\sqrt{R^2 - c^2},$$
$$t = R \pm \sqrt{R^2 - c^2}.$$

From the example, R is 5 and c is 3.

Therefore

$$t = 5 \pm \sqrt{25 - 9}$$
$$= 5 \pm 4 = 9 \text{ or } 1.$$

You will observe that the roots are the distances from both ends of the diameter, or regarded from one end, say A, the chord may be 1 cm. or 9 cm. from that end.

5. The Length of Tangents.

PT is a tangent drawn to a circle of radius R, from a point P, at a distance d from the centre O (fig. 2).

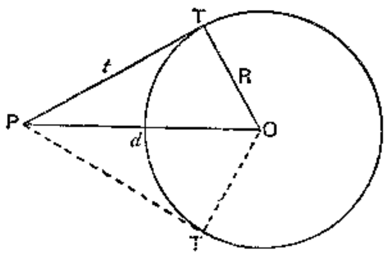


Fig. 2

The angle PTO is a right angle.

Hence $$PT^2 = OP^2 - OT^2$$
$$= d^2 - R^2.$$

If the length of the tangent is t,

$$t^2 = d^2 - R^2$$

or $$t = \pm\sqrt{d^2 - R^2}.$$

Notice that $\cos\angle TOP = \frac{R}{d}$. This enables you to find the angle and the minor or major arc TT.

6. The Horizon.

To an observer at P (fig. 3), T is a point on the horizon which is a circle having its centre on PO (the straight line from P to the centre of the earth).

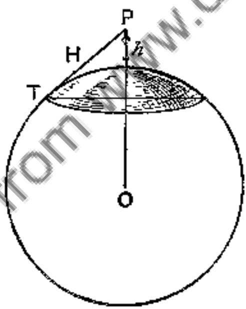


Fig. 3

In calculations upon the horizon, the distance (d) of the observer from the centre of the earth is very little greater than the radius (R) of the earth.

If H is the distance of the horizon from P,

$$H = \sqrt{d^2 - R^2}$$
$$= \sqrt{(d + R)(d - R)}.$$

Now d is practically equal to R, and therefore $d + R$ to 2R, and $d - R$ is the height of P above the surface of the earth.

Call this height h; then

$$H = \sqrt{2Rh}.$$

If R and h are in miles, H will be in miles. It is usual to be given h in feet, in which case $\frac{h}{5280}$ must be placed in the formula instead of h. Substituting 4000 miles for R, the formula becomes:

$$H_{(\text{miles})} = \sqrt{\frac{8000}{5280} h^{(\text{ft.})}}$$
$$= 1{\cdot}231\sqrt{h^{(\text{ft.})}}.$$

EXAMPLE.—Find the distance of the horizon from an observer in a "crow's nest", 100 ft. above the surface of the sea.

$$H = 1{\cdot}231\sqrt{100}$$
$$= 12{\cdot}31 \text{ miles.}$$

EXERCISE XIX (B)

1. The radius of a circle is 6 cm. Find the length of the chord at right angles to a diameter which it intersects 2 cm. from one end.
2. A chord measuring 4 cm. is $1\frac{1}{2}$ cm. from one end of the diameter of a circle which it intersects at right angles. Find the radius.
3. In Exercises 1 and 2, find the angle each chord subtends at the centre of the circle; also the area of the segments into which each chord divides the circle.
4. Divide a straight line 6 in. long into two parts which would contain a rectangle of area 7 sq. in.
5. The cross-sectional area of a stream divided by the perimeter of the wetted part of the channel in which it flows is called the "Hydraulic Mean Depth" of the stream. Find the hydraulic mean depth when water to a depth of 6 in. flows through a pipe of diameter 10 in.
6. In *Robinson Crusoe*, it is stated that Crusoe thought he saw the Peak of Teneriffe from his island. Taking his height above sea level to be 1000 ft. and the Peak to be 12,200 ft. high, calculate approximately how far he was from Teneriffe, assuming his conjectures to be correct.

7. Determine the horizon for an observer in an aeroplane at a height of 10,000 ft.

8. Calculate the distance of the horizon from a person whose eyes are 5 ft. 4 in. above sea level.

CHAPTER XX

THE PROPERTIES OF QUADRATIC EXPRESSIONS AND EQUATIONS, SIMULTANEOUS EQUATIONS, PROBLEMS

1. Quadratic Equations, Quadratic Form, Simultaneous Quadratics.

You have solved quadratic equations; that is, you have found the value of x for which a quadratic expression is equal to 0. Taking the equation $y = ax^2 + bx + c$, when $y = 0$,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

These roots may take various forms. For example:

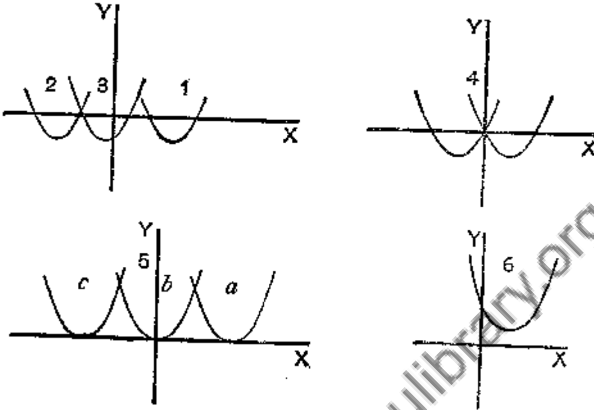
1. They may both be positive, and different.
2. They may both be negative, and different.
3. One may be positive, and the other negative.
4. One may be zero, and the other positive or negative.
5. They may be equal, and be positive, zero or negative.
6. They may be imaginary.

These respective forms are illustrated in figs. 1 to 6.

2. Referring to fig. 7, if A and B are the points at which the graph cuts the axis of x, then

$$\text{AB} = \text{the difference between the roots}$$
$$= \frac{\sqrt{b^2 - 4ac}}{a}.$$

If C is the point at which the graph cuts the axis of y, and if



Figs. 1 to 6

CD is a straight line drawn across the parabola, parallel to the axis of x, then

$$CD = \text{the sum of the roots}$$
$$= \frac{-b}{a}.$$

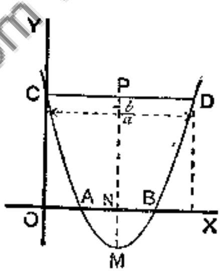


Fig. 7

For, at point C, x is 0, and OC represents the value of ax^2+bx+c when x is 0, i.e. c.

There is, however, another point, namely D, at which the expression is equal to c. To find its co-ordinates, solve the equation

$$ax^2 + bx + c = c, \qquad ax^2 + bx = 0,$$
$$x(ax + b) = 0, \qquad x = 0 \text{ or } \frac{-b}{a}.$$

That is, the co-ordinates of the point D are $x = \frac{-b}{a}$ and $y = c$.

But CD is equal to the x co-ordinate of D, and is therefore equal to $\frac{-b}{a}$.

Hence CD is the sum of the roots.

The position and the direction of CD depend upon the signs of b, c and a. Draw the following graphs, and illustrate this:

(i) $y = 2x^2 - 4x - 6$. (ii) $y = 2x^2 + 4x - 6$.

(iii) $y = 8x^2 + 16x + 6$. (iv) $y = 8x^2 - 16x + 6$.

Even when the roots are imaginary, CD represents their sum.

3. Maximum and Minimum Values.

The vertex of the parabola gives the maximum or the minimum value of the expression.

When the vertex is upwards, the expression has a maximum value; when downwards, a minimum value.

Since the coefficient of x^2 is negative for the former, and positive for the latter, expressions containing $-x^2$ have a maximum value, and those containing $+x^2$ a minimum value.

Also, since the vertex is on the line bisecting CD at right angles (fig. 7), it follows that the value of x for the maximum or the minimum value of the quadratic expression is $\frac{CD}{2}$, i.e. $\frac{-b}{2a}$.

When x has this value, the expression $ax^2 + bx + c$ becomes:

$$\frac{-b^2}{4a} + c, \quad \text{i.e.} \quad \frac{4ac - b^2}{4a}.$$

Example i.—Take the expression $(x^2 - 6x + 4)$.

In this case, $$\frac{-b}{2a} = \frac{-(-6)}{2} = 3.$$

When x is 3, the value of the expression is -5, and this is its minimum value.

Check the result by finding the value of the expression when x is 2, and when x is 4. It will be found that the value when x is 3 is less algebraically than when x is 2 or 4.

Example ii.—Take the expression $(4 + 6x - x^2)$.

Here $$\frac{-b}{2a} = \frac{-6}{-2} = 3.$$

The maximum value is therefore 13.

Check this result as in the last example.

The values of the expressions could have been found directly from the formula $\dfrac{4ac - b^2}{4a}$.

Exercise XX (a)

Draw the graphs of the following expressions, and find graphically the sum of the roots. Verify by calculation.

1. $x^2 + 5x + 6$. 2. $x^2 - 6x + 8$. 3. $-x^2 + 6x - 8$.
4. $-x^2 - 5x - 6$. 5. $x^2 + 6x + 2$. 6. $2x^2 + 2x + 1$.

Say whether the following expressions have maximum or minimum values, and find them in each case. Check your results by trial numbers.

7. $3x^2 + 12x - 2$. 8. $6 + 5x - 2x^2$. 9. $x - x^2$.
10. $4 - 3x^2$. 11. $3 + 2x^2$.
12. $2x^2 + 3x + 5$. (Notice that the roots of the equation $2x^2 + 3x + 5 = 0$ are imaginary.)
13. Applying the same reasoning to $3 - 4x + 0x^2$, what can you say about its values?
14. If the path of a projectile is represented by the equation $y = -\frac{1}{18}x^2 + \frac{2}{3}x$, what is its form? If x and y are in miles, determine
    (i) the greatest height to which the projectile rises, i.e. the maximum value of y;
    (ii) the horizontal range of the projectile, i.e. the distance between the values of x for which y is 0.
15. If the path of a projectile is given by the equation $y = 0{\cdot}4x - 0{\cdot}04x^2$, find (i) the horizontal range; (ii) the greatest height to which the projectile rises. (x and y are in miles.)
16. Without solving the equations, state whether the following have real or imaginary roots:

(i) $x^2 + x + 3 = 0$. (ii) $x^2 + x = 3$.
(iii) $2x^2 + 3x + 2 = 0$. (iv) $5 - 2x - x^2 = 0$.
(v) $4 - 2x^2 = 5x$. (vi) $3x^2 - 4x + 2 = 2x$.

17. The minimum value of a quadratic expression is 6, 3 being the corresponding value of x, and when x is 0 the value is 21. Find the expression and the roots of the equation, expression = 0.

18. When x is 0, the value of a quadratic expression is -48. When x is 3 the expression has its minimum value, namely -75. Find the expression.

19. Find the value of y in terms of x from the equation

$$3x^2 + 2xy + y^2 = 1,$$

and say for what values of x, y is imaginary.

Graph a few of the real values of y.

20. If α and β are the roots of the equation $ax^2 + bx + c = 0$, show that $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.

21. If α and β are the roots of the equation $3x^2 - 5x + 1 = 0$, find the equation whose roots are $\dfrac{\alpha}{1 + \beta}$ and $\dfrac{\beta}{1 + \alpha}$.

4. Quadratic Form.

Any equation which has the form $ax^{2n} + bx^n + c = 0$ can be solved by the methods given.

Example.—$2x^4 - 3x^2 - 36 = 68$ (note that x^4 is the square of x^2),

$$x^4 - \tfrac{3}{2}x^2 = \tfrac{104}{2}.$$

Completing the square,

$$x^4 - \tfrac{3}{2}x^2 + (\tfrac{3}{4})^2 = \tfrac{104}{2} + \tfrac{9}{16} = \tfrac{841}{16};$$

$$\therefore\ x^2 - \tfrac{3}{4} = \pm\tfrac{29}{4}$$

and

$$\therefore\ x^2 = \tfrac{3}{4} \pm \tfrac{29}{4}$$
$$= 8 \text{ or } -\tfrac{26}{4}.$$

x is now found by taking the square root of 8. The other root is, of course, imaginary in this case.

$$\therefore\ x = \pm 2\sqrt{2}.$$

Exercise XX (b)

Solve:

1. $x^4 - 20x^2 + 64 = 0$.
2. $x^4 - 13x^2 + 40 = 4$.
3. $ax^4 + bx^2 + c = 0$.
4. $x^6 - 14x^3 = -48$.

5. Draw the graph of $x^4 - 13x^2 + 36$, and determine how many times it cuts the axis of x, and also, how many turns or vertices it has.

6. By the method of testing factors (p. 172), show that a is one root of the equation $x^3 - 3a^2x + 2a^3 = 0$. Hence find the remaining roots.

7. Graph the expression $x^3 - x + 2$, and find how many times it cuts the axis of x, and how many vertices it has.

5. Simultaneous Equations containing one or more Unknowns to the Second Power.

TYPE I.—*Solved by substitution.*

(a)
$$5x + 2y = -4, \quad \text{(i)}$$
$$y = x^2 + 3x + 5. \quad \text{(ii)}$$

Equation (i) contains the first power of x, only.

From (i),
$$2y = -4 - 5x,$$
$$y = \frac{-4 - 5x}{2}. \quad \text{(iii)}$$

Substituting this value in equation (ii),
$$\frac{-4 - 5x}{2} = x^2 + 3x + 5,$$
$$-4 - 5x = 2x^2 + 6x + 10,$$
$$2x^2 + 11x + 14 = 0.$$

This is a quadratic equation, the roots of which are:
$$x = \frac{-11 \pm \sqrt{121 - 112}}{4}$$
$$= -\tfrac{7}{2} \text{ or } -2.$$

From (iii),
$$y = \frac{-4 - 5x}{2},$$
$$y = \tfrac{27}{4} \text{ or } 3.$$

Note carefully that a solution of the equations consists of a value of x *and* a corresponding value of y. In this example there are two solutions, viz.

$$(x = -\tfrac{7}{2}, \quad y = \tfrac{27}{4})$$
and
$$(x = -2, \quad y = 3).$$

(*b*) $$3x + 2y = 12, \qquad \text{(i)}$$
$$2x^2 - 3y^2 = -19. \qquad \text{(ii)}$$

From (i), $$y = \frac{12 - 3x}{2}. \qquad \text{(iii)}$$

Substituting in (ii),

$$2x^2 - 3\left(\frac{12 - 3x}{2}\right)^2 = -19,$$

$$19x^2 - 216x + 356 = 0.$$

$$x = 2 \quad \text{or} \quad 9\tfrac{7}{19}.$$

y is found from equation (iii). The solutions are

$$(x = 2,\ y = 3)$$

and $$(x = 9\tfrac{7}{19},\ y = -8\tfrac{1}{19}).$$

(*c*) $$x + y = a, \qquad \text{(i)}$$
$$xy = b. \qquad \text{(ii)}$$

From (i), $$y = a - x. \qquad \text{(iii)}$$

Substituting in (ii), $$x(a - x) = b. \qquad \text{(iv)}$$

Equation (iv) is an ordinary quadratic.

The rest is easy.

TYPE II.—*All terms of the second degree. Solved by finding the ratio of the unknowns.*

(*a*) $$x^2 + y^2 = 164, \qquad \text{(i)}$$
$$xy = 80. \qquad \text{(ii)}$$

Let $y = kx$; then, from (i),

$$x^2 + k^2x^2 = 164; \qquad \text{(iii)}$$

from (ii), $$kx^2 = 80. \qquad \text{(iv)}$$

Dividing (iii) by (iv) and cancelling x^2,

$$\frac{1 + k^2}{k} = \frac{164}{80}.$$

Cross multiplying, $$80 + 80k^2 = 164k,$$

$$80k^2 - 164k + 80 = 0,$$

or $$20k^2 - 41k + 20 = 0,$$

from which $$k = \frac{41 \pm \sqrt{(41)^2 - 1600}}{40}$$

$$= \tfrac{5}{4} \quad \text{or} \quad \tfrac{4}{5}.$$

From (iv), x is found. Then from (ii), y is found.
There are four solutions:

$$(x = 10, y = 8),\ (x = -10, y = -8),$$
$$(x = 8, y = 10),\ (x = -8, y = -10).$$

(*b*)
$$2x^2 + 3xy + y^2 = 70,$$
$$6x^2 + xy - y^2 = 50.$$

Let $y = kx$, and proceed as in Example (*a*).

These equations can be solved graphically, by plotting the values of y against values of x, and finding the intersection points of the graphs. It will be found that the graph of the equation $x^2 + y^2 = a^2$, where a is a number, is a circle whose centre is the origin and whose radius is a (see equation, Type II (*a*)).

6. Application to Surds.

From the identity $(\sqrt{x} \pm \sqrt{y})^2 = x + y \pm 2\sqrt{xy}$, the square root of an expression of the type $N \pm 2\sqrt{M}$ can be readily found; for it follows that

$$\text{(i) } x + y = N. \qquad \text{(ii) } xy = M.$$

From these simultaneous equations x and y can be determined.

Example.—Find the square root of $20 - 8\sqrt{6}$.

$$20 - 8\sqrt{6} = 20 - 2\sqrt{96}.$$

Hence, if the root is of the form $\sqrt{x} - \sqrt{y}$,

$$\text{(i) } x + y = 20. \qquad \text{(ii) } xy = 96.$$

The values of x and y can be found from these simultaneous equations, but they are obviously 12 and 8. The square root is therefore $\pm(\sqrt{12} - \sqrt{8})$, which when simplified becomes,

$$\pm(2\sqrt{3} - 2\sqrt{2}) = \pm 2(\sqrt{3} - \sqrt{2}).$$

Exercise XX (c)

1. Solve the example Type II (*a*) by forming two equations, one by adding and the other by subtracting twice equation (ii) from equation (i), and then extracting the square root of the resulting equations.

Solve:

2. $2x - 5y = 3$ and $x^2 + xy = 20$. 3. $x - y = 3$, $x^2 + y^2 = 65$.

4. $\frac{x}{12} + \frac{y}{10} = x - y,\ \frac{7xy}{15} - \frac{2x}{3} = 2y.$

5. $9y - 2 = xy,\ xy + 2 = x.$

6. $x^2 + y^2 = 20,\ x + y = xy - 2.$

7. $x^2 + xy = 21 + 6y^2,\ xy - 2y^2 = 4.$

Find graphically the values of x for which the following expressions have the same values. Find these values.

8. $2x - 5$ and $x^2 - 3x + 1$.

9. $x^2 + 3x + 6$ and $-2x^2 - 9x - 3$.

10. $x^2 + 6x + 5$ and $12 + x - x^2$.

11. $\frac{1}{x}$ and $x^2 - x - 6$.

Solve:

12. $x^2 + y^2 = 5,\ 3x + 4y = 2.$

13. $\frac{1}{x} + \frac{1}{y} = 1,\ 6x + 2y = 1.$

14. $2x - y = 3,\ 2x^2 + xy = 2.$

15. $3(x - 1)^2 + (y + 2)^2 = 9,\ x + y = 1.$

16. (i) $x^3 + y^3 = 189,$ $x^2 - xy + y^2 = 21.$ (ii) $3x^2 - 2xy - y^2 = 35,$ $2x^2 + xy - 3y^2 = 0.$

17. $x + y = 5{\cdot}17,\ x^2 + y^2 = 14{\cdot}25.$

Find the square root of:

18. (i) $9 + 2\sqrt{14}$, (ii) $9 - 2\sqrt{14}$.

19. $23 - 6\sqrt{10}$.

20. $5 - 2\sqrt{6}$.

21. $36 + \sqrt{1292}$.

22. $2(a + \sqrt{a^2 - b^2})$.

23. $\frac{a^2 + b^2}{b(a - b)} - \sqrt{\frac{4a(a + b)}{b(a - b)}}$.

24. The difference between two numbers is 4, and the sum of their squares 106. Find them.

25. Divide 20 into two parts such that the sum of the squares of the parts is less by 8 than forty times one of the parts.

26. The perimeter of a rectangle is 34 in., and its area 60 sq. in.; find the length of its diagonals.

27. Find a sector such that if its radius were increased by 5 in. its area would increase in the ratio of 3 to 2.

28. The distance in feet through which a body falls in a time t sec. is given by the equation

$$d = 16t^2.$$

When a stone is dropped down the shaft of a mine the sound of its impact with the bottom is heard at the surface 10 sec. after the release of the stone.

If sound travels at the rate of 1100 ft. per second, find the depth of the shaft.

29. Two men start at the same time for a town 75 miles distant, one cycling and the other by motor car. If the motorist travels 10 miles an hour faster than the cyclist and reaches the town 2 hr. 40 min. before him, find the rate at which each travels.

30. Draw and examine the graphs of $y = 3 \pm \sqrt{4 - x^2}$.

Find (i) the range of x within which there are two values of y for a value of x.

(ii) The values of x for which there is only one value of y.

(iii) The ranges of the values of x for which there are no real values of y.

31. Draw and examine the graph of $\frac{x^2}{36} + \frac{y^2}{9} = 1$.

What change would you make in this equation so that the graph would be the circumference of a circle?

Revision Exercise II

1. What is the error per cent in using: (i) $\sqrt{2} + \sqrt{3}$ for π; (ii) $\frac{1 + \sqrt{3}}{\sqrt{3}}$ for $\frac{\pi}{2}$?
2. Find the equation to the straight line passing through the points $(-4, 13)$ and $(5, -5)$.
3. What is the equation to the straight line which cuts the axis of y at -3, and makes an angle of 30° with the axis x, the scales of the axes being alike?
4. Divide 75 into two parts such that one part is 9 more than twice the other.
5. If $E = C\sqrt{R^2 + p^2L^2}$, find C when $E = 161$, $R = 10$, $p = 40\pi$ and $L = 0\cdot1$.
6. (1) What factors of $x^4 - 5x^2 + 4$ are factors of $x^5 - 11x + 10$ also?
   (2) Factorize: (i) $x^4 - 6x^2y^2 - 16y^4$; (ii) $a^4 + 2a^3b - 2ab^3 - b^4$;
   (iii) $2x^3 - 3x^2y - 2x + 3y$; (iv) $2x^4 + 24x - 8x^2 - 48$.
7. Without multiplying the whole expressions, find the coefficient of x^2 in the product of $(x^3 - 3x + 2)$ and $(2x^2 + 5x - 3)$.
8. Simplify: (i) $3a\sqrt{2} - 2b\sqrt{3} + 2\sqrt{3} - 3\sqrt{2} + 5b\sqrt{3} + a\sqrt{2}$, and find its value when $a = 2$ and $b = -3$.

   (ii) $\frac{x^6 - y^6}{(x-y)(x^2 - xy + y^2)}$.

9. A number is exactly divisible by 3, if the sum of its digits is exactly divisible by 3. Prove this for a number of three digits, taking a, b and c to represent the digits. [Hint: $100a = (99a + a)$.]

10. Show that the cube of the sum of two numbers is equal to the sum of their cubes, together with three times their sum multiplied by their product.

11. Compute: $\pi \dfrac{(15{\cdot}83)^2 \times 0{\cdot}0385}{\sqrt[3]{10}} + 15{\cdot}83\pi \dfrac{26{\cdot}4\sqrt{5{\cdot}37}}{0{\cdot}852}$.

12. On the same diagram draw the graphs of $2x^2$ and $12 - 5x$, and find the values of x for which $2x^2 = 12 - 5x$.

Apply what you have learnt to solve the equation $x^3 - 7x + 6 = 0$.

13. If $h = \frac{1}{2}gt^2$ and $v = gt$, show that $h = \dfrac{v^2}{2g}$.

14. Determine the equation to the parabola passing through the points, $(1, 8)$, $(-2, -7)$, and $(2, 5)$. Find also the co-ordinates of its vertex.

15. Show that the least value of $3x^2 - 6x + 5$ is 2.

16. Show that $\log(x + \sqrt{x^2 - 1}) = -\log(x - \sqrt{x^2 - 1})$. See Ex. XV(B), 48.

CHAPTER XXI

GEOMETRY AND TRIGONOMETRY

1. Relations between the Sides and Angles of any Triangle.

1. Let ABC be the triangle, the lengths of the sides being a, b, c named according to the opposite angle as shown.

There are two cases to be considered, namely, one in which the angle, say C, from which the triangle is considered is acute (fig. 1) and the other in which the angle is obtuse (fig. 2).

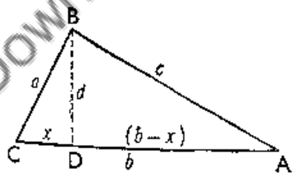


Fig. 1

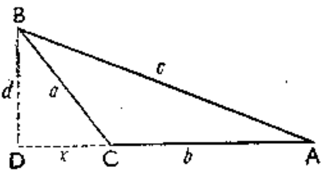


Fig. 2

From B draw BD ⊥ to the opposite side (fig. 1), or the side produced (fig. 2).

Let BD = d and CD = x, then in fig. 1, AD = $(b - x)$ and in fig. 2, AD = $(b + x)$.

In fig. 1 $c^2 = d^2 + (b - x)^2$ since ∠ADB is a right angle

$= a^2 - x^2 + b^2 + x^2 - 2bx$ since $d^2 = a^2 - x^2$

$$= a^2 + b^2 - 2bx. \qquad \ldots \qquad (1)$$

In fig. 2 $c^2 = d^2 + (b + x)^2$

$= a^2 - x^2 + b^2 + x^2 + 2bx$

$$= a^2 + b^2 + 2bx. \qquad \ldots \qquad (2)$$

Now in fig. 1, $\frac{x}{a} = \cos C$, from which $x = a \cos C$. Therefore, *when C is acute*, equation (1) gives

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

The case when C is obtuse needs further consideration. So far, we have given no meaning to sin A, cos A, tan A, and the other ratios of the angle A except for the case when A is an acute angle. In a later chapter (Chap. XXIII, p. 270) we shall define sin A, cos A, etc., when A is an angle of any magnitude. Meanwhile, for an obtuse angle A, we give the definitions

$$\left.\begin{aligned} \sin \mathbf{A} &= \sin (180° - \mathbf{A}) \\ \cos \mathbf{A} &= -\cos (180° - \mathbf{A}) \end{aligned}\right\} \qquad \ldots \qquad (3)$$

For example,

$$\cos 130° = -\cos(180° - 130°) = -\cos 50° = -{\cdot}6428.$$

In fig. 2 we therefore have

$$x = a \cos(180° - C)$$
$$= -a \cos C.$$

Equation (2) becomes, for C obtuse,

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

which is the same equation as the one already found for C acute. It will be noticed that the formula gives the third side when two sides and the included angle are given.

Considering each of the angles in turn, we have the three important equations:

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A.}$$
$$\boldsymbol{b^2 = c^2 + a^2 - 2ca \cos B.}$$
$$\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos C.}$$

EXAMPLES.—Two sides of a triangle are 3 in. and 4 in. respectively. Find the remaining side when the included angle is (i) 50°; (ii) 90°; (iii) 130°.

$$\text{(i)}\quad c^2 = a^2 + b^2 - 2ab\cos C$$
$$= 3^2 + 4^2 - 2 \times 3 \times 4\cos 50^\circ$$
$$= 9 + 16 - 24 \times \cdot 6428$$
$$= 25 - 15\cdot 4272$$
$$= 9\cdot 5728.$$
$$\therefore\ c = \sqrt{9\cdot 5728} = 3\cdot 09 \text{ in.}$$

$$\text{(ii)}\quad c^2 = 9 + 16 - 24\cos 90^\circ$$
$$= 25 - 0, \quad \text{since } \cos 90^\circ = 0,$$
$$\therefore\ c = \sqrt{25} = 5 \text{ in.}$$

$$\text{(iii)}\quad c^2 = 3^2 + 4^2 - 24\cos 130^\circ$$
$$= 25 - 24 \times -\cos 50$$
$$= 25 - 24 \times -\cdot 6428$$
$$= 25 + 15\cdot 4272$$
$$= 40\cdot 4272.$$
$$\therefore\ c = \sqrt{40\cdot 4272} = 6\cdot 358 \text{ in.}$$

2. Referring to fig. 1,

since $\qquad CD = a\cos C$

and $\qquad AD = c\cos A$

$\therefore$ by addition $\quad AC = c\cos A + a\cos C,$

i.e. $\qquad \boldsymbol{b = c\cos A + a\cos C.}$

Similarly, $\qquad \boldsymbol{c = a\cos B + b\cos A,}$

$\qquad \boldsymbol{a = b\cos C + c\cos B.}$

It is easily proved by using fig. 2, and the definition $\cos C = -\cos(180^\circ - C)$ (p. 230), that these relations still hold good when one of the angles is obtuse.

2. Area of a triangle in terms of its sides.

Given $\triangle ABC$ whose sides are a, b, c units long (fig. 3). Find its area in terms of a, b, c.

Preliminary: If s is the semi-perimeter, then

$$2s = a + b + c.$$

Subtract $$2c = \qquad 2c,$$

then, $$\overline{2(s - c)} = \overline{a + b - c}.$$

Similarly, $$2(s - b) = c + a - b$$

and $$2(s - a) = b + c - a.$$

These relations will be useful.

Draw the altitude line BD, and let BD $= d$ and CD $= x$. (We can take C to be acute.)

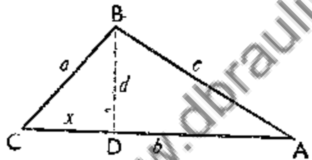


Fig. 3

Now $$c^2 = a^2 + b^2 - 2bx;$$

$$\therefore\ x = \frac{a^2 + b^2 - c^2}{2b}.$$

Again, $$d^2 = a^2 - x^2 = (a + x)(a - x)$$

$$= \left(a + \frac{a^2 + b^2 - c^2}{2b}\right)\left(a - \frac{a^2 + b^2 - c^2}{2b}\right)$$

$$= \left\{\frac{2ab + (a^2 + b^2 - c^2)}{2b}\right\}\left\{\frac{2ab - (a^2 + b^2 - c^2)}{2b}\right\}$$

$$= \frac{\{(a + b)^2 - c^2\}\{c^2 - (a - b)^2\}}{4b^2}.$$

By factors $$= \frac{(a + b + c)(a + b - c)(c + a - b)(c - a + b)}{4b^2}$$

$$= \frac{2s \times 2(s - c) \times 2(s - b) \times 2(s - a)}{4b^2}.$$

From which $$d = \frac{2}{b}\sqrt{s(s - a)(s - b)(s - c)}.$$

Now area of

$$\triangle ABC = \tfrac{1}{2}bd$$
$$= \tfrac{1}{2}b \times \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

EXAMPLE.—If the sides are 12, 16 and 20 cm.,

$$s = \frac{12 + 16 + 20}{2} = 24.$$

$$\text{Area} = \sqrt{24 \times 12 \times 8 \times 4} = 96 \text{ sq. cm.}$$

The discovery of this useful formula is attributed to Hero of Alexandria about 80 B.C.

EXERCISE XXI

1. Arrange the three equations (p. 230) for finding cos A, cos B and cos C respectively.
2. Two sides of a triangle are 7 cm. and 10 cm., and the included angle is 65°. Find the third side and the remaining angles.
3. Find the angles of the triangle whose sides are 8, 10, and 12 cm., and of the triangle whose sides are 8, 12 and 16 cm.
4. Find the area of the triangles of Exercise 3.
5. What does Hero's formula become if a, b and c are all equal as in the case of an equilateral $\triangle$ of, say, side a?
6. Hero's formula can be used to find the altitudes of a triangle of given sides. There are three altitudes, one for each vertex; find them for the triangle whose sides are 8, 16 and 20 cm.

CHAPTER XXII

AREA BOUNDED BY A GRAPH, APPLICATIONS TO MENSURATION, AND SCIENCE

1. Graphs.

The area bounded by a graph, the axis of x and two ordinates, has a definite significance.

To understand it, consider a graph parallel to the axis of x (fig. 1). The equation to the graph is $y = 0x + 5$.

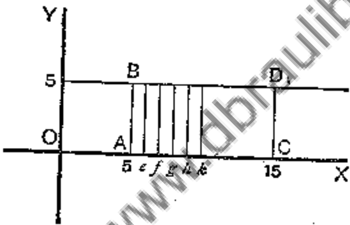


Fig. 1

Draw ordinates AB and CD at $x = 5$ and at $x = 15$; then the rectangle ABDC has a length, AC, of 10 units and an altitude of 5 units.

The area of ABDC is 50 units, the unit being the square on the unit of length.

The area thus represents the product of 10 and 5.

The figure ABDC can be regarded as consisting of a number of narrow strips, standing on the small bases Ae, ef, fg, gh, etc., and having the same altitude, namely AB.

Then, since the area ABDC is the sum of the areas of these strips,

$$\begin{aligned} \text{ABDC} &= \text{AB} \times \text{A}e + \text{AB} \times ef + \text{AB} \,.\, fg + \text{etc.} \\ &= \text{AB}(\text{A}e + ef + fg + \text{etc.}) \\ &= \text{AB} \,.\, \text{AC}. \end{aligned}$$

Now this is true no matter how small the parts on the axis of x may be.

The same is true in Arithmetic; for example, $3 \times 6 = 3(3 + 2 + 1)$, or 3(any set of numbers the sum of which is 6). An area then may represent a product.

2. Take a straight-line graph inclined to the axis of x. Consider the area between two ordinates, AB and CD (fig. 2). In this case, not only does x change from OA to OC, but y also, from AB to CD.

The area of the trapezoid ABDC may be regarded as made up of narrow strips, the strips being so narrow that they are practically rectangles. It will be noticed, however, that the altitude of the strips increases from AB to CD.

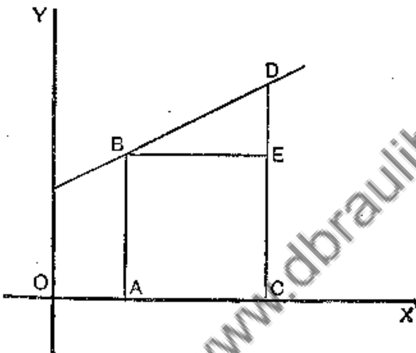


Fig. 2

The area is the sum of these small areas, and as before represents a product, but, in this case, of numbers which are changing.

The product of x and y, when x changes from OA to OC and y changes from AB to CD, is represented by the area of the trapezoid ABDC, which by Mensuration is:

$$AC \times \frac{AB + CD}{2} = AC \times \frac{AB + CE + ED}{2}$$

$$= AC \times \frac{2AB + ED}{2} = AC\left(AB + \frac{ED}{2}\right).$$

The following examples show the importance of these results.

Applications.

(i) A body moves with a uniform speed of 16 ft. per second. Construct a graph showing the relation between speed and time, and from it find the distance travelled in 20 sec.

Represent time on the axis of x, and the speed in feet per second on the axis of y (fig. 3). The graph is, of course, parallel to the axis of x.

Now, distance is equal to the product of speed and time, and therefore if an ordinate is drawn from 20 on the axis of x, the area bounded by this ordinate, the graph, the axis of y and the axis of x will represent the distance travelled in 20 sec.

The distance is $16 \times 20 = 320$ ft.

From the two diagrams, it will be seen that this result does not depend upon the scales of the axes.

(ii) The work done by a force is measured by the product of

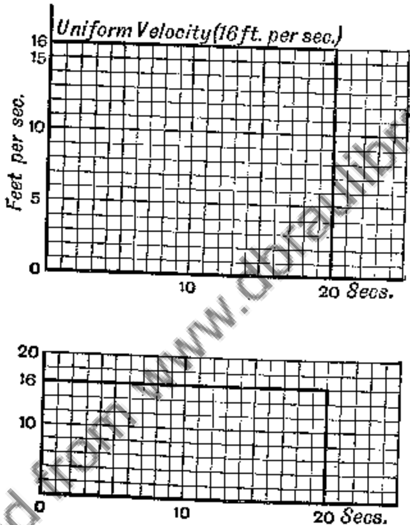


Fig. 3

the displacement and the force acting in the direction of the displacement. Show by a graph the work done when a force of 20 lb. produces a displacement 12 ft. in the direction of the force. This graph, like that in application (i), is a rectangle.

(iii) A body with an initial velocity of 10 ft. per second gains velocity at the rate of 2 ft. per second every second. Construct a graph showing the relation between velocity and time. From it find the relation between velocity and time, and also the distance travelled in 6 sec. (fig. 4).

The gradient of the graph is 2, and the added constant 10.

The equation is therefore of the form $y = 2x + 10$.

Or, if we call the velocity v and the time t, the equation becomes $v = 2t + 10$.

When t is 6 sec., v is $(2 \times 6 + 10) = 22$ ft. per sec.

The ordinate at the point 6 on the axis of time represents 22 ft. per second.

Draw this ordinate. Then, since distance is obtained by multiplying velocity by time, the area OCDt represents the distance travelled in the interval 0 to 6 sec.

$$\text{Therefore the distance} = \frac{10 + 22}{2} \times 6 = 96 \text{ ft.}$$

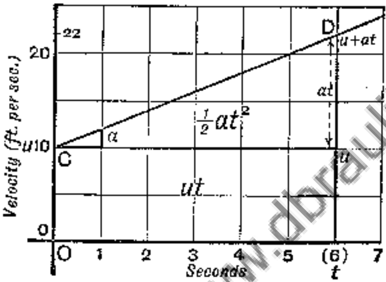


Fig. 4

Observe that:

(i) The gradient of the graph gives the rate at which the velocity changes. (Rate of change of velocity is called *acceleration.*)

(ii) The added constant is the initial velocity.

It is now easy to deduce the general formula for the distance travelled in a given interval of time by a body moving with uniform acceleration (fig. 4).

Let u = the initial velocity, a = the acceleration, i.e. the change in velocity in unit time, and t = the units in the interval of time, reckoned from the instant at which the velocity was u.

Then a is the gradient of the graph, u the added constant, and the velocity v at the end of the interval of time, $u + at$. (at is the total change in velocity.)

Drawing the ordinate at point t, its length represents $(u + at)$.

Hence the distance (d) travelled in the interval 0 to t is represented by the area OCDt.

$$\text{Therefore,} \quad d = \frac{\text{OC} + t\text{D}}{2} \times \text{O}t = \frac{u + (u + at)}{2} \times t$$

$$= ut + \tfrac{1}{2}at^2.$$

Exercise XXII (A)

Special Cases and Examples

1. If the body starts from rest, $u = 0$. Draw the graph for this case, and by it determine the formula for the distance covered in an interval 0 to t.

2. If the speed of the body is decreasing, the graph has a down gradient, i.e. the coefficient of t is negative.

   Draw a graph for this case, and deduce the formula for distance.

3. A falling body has a uniform acceleration of +32 ft. per sec. every second (approx.). Draw a graph showing the relation between velocity and time (the body starting from rest), and determine the distance covered in the following intervals of time:

   (i) During the first second.
   (ii) During the first 6 sec.
   (iii) During the interval from the beginning of the fourth to the end of the ninth second.

4. Show on the general graph (fig. 4) that the average velocity during the interval 0 to t is the same as the actual velocity at the middle of the interval.

5. When a body is projected vertically from the earth it loses speed at the rate of 32 ft. per second every second. If the velocity of projection is 100 ft. per second, draw a graph showing how speed changes with time. From it find the formula for calculating the height to which the body will rise, and also the time taken.

   Complete the graph to represent the return journey also.

6. *Work done by a Variable Force.*

   A wire rope weighing 10 lb. per foot is coiled on the floor. It is gradually lifted vertically by taking hold of one end and raising it through a vertical height equal to the length of the rope. If the length is 60 ft., find the work done when the lower end just leaves the floor.

   Plot the graph showing the force when the upper end is at various distances from the floor.

   The work done is represented by the area of the triangle.

6 (*continued*). Find also

(i) The work done in lifting the first half of the rope.

(ii) The work done in lifting the remaining half of the rope.

7. The following numbers were obtained in stretching a spring by hanging weights at its free end:

Increase in length	0″	0·2″	0·4″	0·8″	1·0″	5·0″
Weight applied -	0	1 lb.	2 lb.	4 lb.	5 lb.	25 lb.

Draw a graph and find:

(i) An equation connecting the force and the elongation.

(ii) The work done in stretching the spring 5 in.

(iii) If the length of the unloaded spring is 2 ft., an equation connecting the length with the force applied.

8. The following numbers give the current in a short-circuited armature for different field currents:

Field current	0	0·075	0·163	0·253	0·342	x
Arm. current	14	45·4	85·5	125·6	165·6	y

Find the equation connecting armature current and field current.

3. The relation between the graph of "velocity", and that of "positions" of a body moving with uniform acceleration is specially interesting.

The graphs are shown one directly under the other in fig. 5. The initial velocity is taken as 5 ft. per sec. and the acceleration as 2 ft. per sec. per second.

In the upper graph, area AO1B represents the distance, 6 ft., covered in 1 sec. In the lower graph, length 1A represents the distance covered in 1 sec. Similarly areas AO2C, AO3D, AO4E represent the distances covered in 2, 3 and 4 sec. respectively, and in the lower graph, these distances are represented respectively by the lengths 2B, 3C, 4D. Thus it is seen that the ordinates of the graph of positions represent the areas of the graph of velocity.

It will be readily understood that the difference between the ordinates 3C and 2B is equivalent to the area C23D.

The lower graph is often called the **integral** of the upper graph.

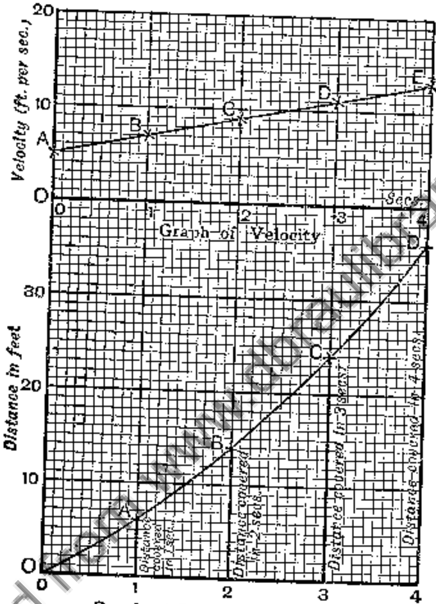


Fig. 5

4. Area bounded by the Parabolic Graph.

In Mensuration you have learnt that the volume of a prism is obtained by multiplying the area of the base by the height. There is another way of expressing this rule.

Take a square prism and imagine a number of sections all parallel to the base to be made. These sections, called *right sections*, have the same shape and area as the base. Observe that they are perpendicular to the line of altitude (fig. 6).

The rule for volume may be expressed as the product of the altitude and the area of the section perpendicular to the altitude.

If we plot the area of section and the altitude at which the section is made for a square prism of altitude 8 in. and side of base 3 in. (fig. 7), the graph is a horizontal straight line. The area enclosed by the graph, ordinates at points 0 and 8 on the axis of x, and by the axis of x or altitude, represents the product of section and altitude, and therefore the **volume** of the prism.

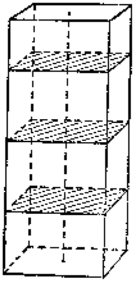


Fig. 6

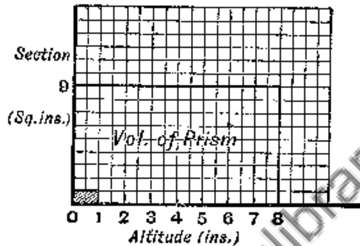


Fig. 7

In this case, the area representing the volume is a rectangle (8 × 9). The volume is therefore 72 c. in.

If we take a cylinder (fig. 8), the right sections are again equal to the area of the base. The graph of sections is therefore a horizontal straight line. Again, the area representing the volume is a rectangle.

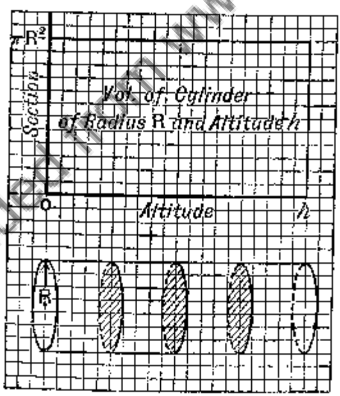


Fig. 8

5. Consider next a square pyramid. Like the square prism, its right sections are squares, but they increase in area from the apex to the base.

Now, it is known that the length of the edge of a section is proportional to its distance from the apex. Thus, taking a pyramid of base 12 in. edge, and altitude 8 in., the edge and area of section at distances from the apex are as follows:

Distance from Apex	Edge of Section	Area of Section
0 in.	0 in.	0 sq. in.
2 in.	3 in.	9 sq. in.
4 in.	6 in.	36 sq. in.
6 in.	9 in.	81 sq. in.
8 in.	12 in.	144 sq. in.

Plot the area of section against distance from the apex. The graph is a parabola. The area bounded by the graph represents the volume of the pyramid (fig. 9).

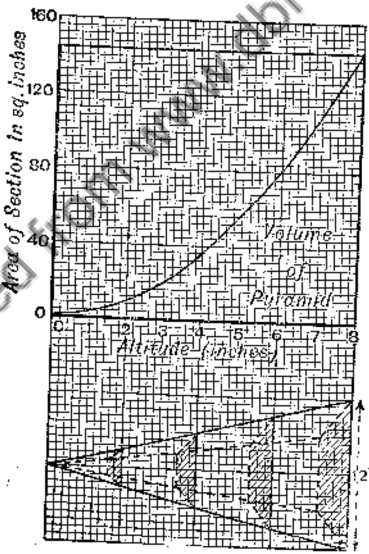


Fig. 9

A more advanced student would be able to calculate this area from the equation to the graph. It can be obtained, however, by counting the squares, or, better, by cutting out the figure in metal or cardboard, and comparing its weight with that of a sheet of known area. The result is 384 units, and represents 384 c. in.

On the same axes represent the volume of the prism having the same base and altitude, and compare the volumes. You will find that the volume of the pyramid is one-third that of the prism.

It is worth remembering that the area enclosed by the parabola, axis of x, and the end ordinate is one-third of the rectangle having this portion of the axis of x as length and this ordinate as height. Observe that the vertex of the parabola is at the origin. (See p. 202.)

6. In the same manner the volume of a cone can be compared with the volume of the cylinder having the same base and altitude. The right sections are circles of increasing radii, any radius being proportional to its distance from the apex.

Take a cone of, say, height 8 cm. and diameter 12 cm.

Tabulate the areas of sections at different distances from the apex, thus:

Distance from Apex	Section		
	Diameter	Radius	Area
0 cm.	0 cm.	0 cm.	0 sq. cm.
2 cm.	3 cm.	1·5 cm.	$2{\cdot}25\pi$ sq. cm.
4 cm.	6 cm.	3 cm.	9π sq. cm.
6 cm.	9 cm.	4·5 cm.	$20{\cdot}25\pi$ sq. cm.
8 cm.	12 cm.	6 cm.	36π sq. cm.

In plotting the numbers there is no need to substitute the actual value of π. Scale the axis of area (y) in terms of π as shown in fig. 10.

You will find on plotting the area of section against the distance from the apex, that the graph is a parabola. On the same axes, draw the graph for the corresponding cylinder. The areas bounded by these graphs to the ordinate at $x = 8$ represent the volumes of the cone and the cylinder respectively. On determining these, you will find that the volume of the cone is one-third that of the cylinder,

$$\text{i.e. vol. of cone} = \frac{36\pi \times 8}{3} = 96\pi \text{ c.c.}$$

Hence the rule:

The volume of a cone is one-third the product of the area of the base and the altitude.

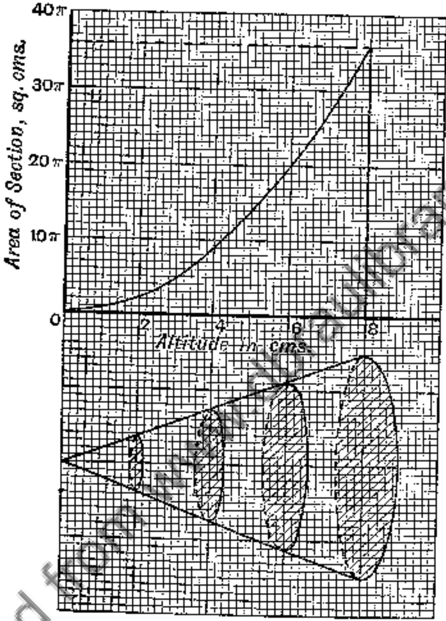


Fig. 10

7. Frustum of a Cone.

The frustum of a cone is the part remaining when a cone is cut off a larger cone by a plane parallel to the base.

In fig. 11, r = radius of the top,

R = ,, ,, bottom,

h = height of the frustum,

x = altitude of the cone cut off,

then since $\triangle ABC$ and $\triangle CEF$ are similar,

$$\frac{x}{r} = \frac{h}{R - r}, \text{ from which } x = \frac{rh}{R - r}.$$

This enables you to find the height of the cone cut off in terms of the dimensions of the frustum.

The volume of the frustum is the difference between the full cone and the part cone, i.e. $\frac{1}{3}\pi R^2(x + h) - \frac{1}{3}\pi r^2 x$.

Example.—Find the volume of the frustum of a cone, height 5 in., radius of top 4 in., radius of bottom 6 in.

Here $\quad \frac{x}{4} = \frac{5}{6 - 4}, \quad$ from which $\quad x = 10$ in.

The height of the full cone is $10 + 5 = 15$ in.

$$\text{Vol. of frustum} = \tfrac{1}{3}\pi 6^2 \times 15 - \tfrac{1}{3}\pi 4^2 \times 10$$
$$= 126\tfrac{2}{3}\pi \text{ c. in.}$$
$$= 398 \text{ c. in. (approx.).}$$

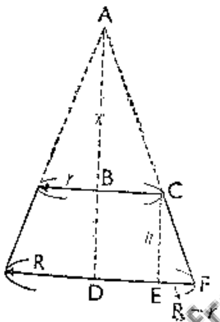


Fig. 11

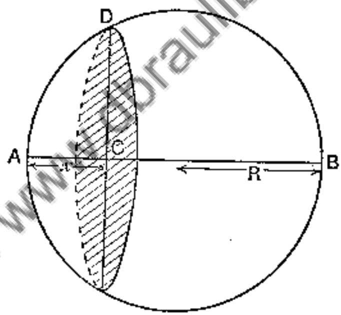


Fig. 12

8. The Sphere.

If sections are made at right angles to a diameter of a sphere, a number of parallel circles is obtained.

The areas of these circles vary.

Referring to fig. 12, consider the section at a point C a distance x from the end A of the diameter AB.

Let r be the radius of this section, and R the radius of the sphere.

Then CD $= r$ and CB $= (2R - x)$.

It has been established that $(CD)^2 = AC \cdot CB$,

$$\text{i.e. } r^2 = x(2R - x).$$

Now the area of section is πr^2, which, by the above, is equal to:

$$\pi x(2R - x) = 2\pi Rx - \pi x^2.$$

Since this expression contains x^2, we conclude that the graph of section and distance is a parabola. Moreover, since the sign of x^2 is negative, and the expression contains x, the vertex is upwards and situated to the right of the axis of y.

Plot the graph for various values of x, i.e. for points along AB, and the area bounded by it will represent the volume of the sphere.

Take a sphere of, say, 10 cm. diameter.

Tabulate as follows:

x, cm.	(Radius of Section)^2 =product of segments of diameter	Area of Section, sq. cm.
0	$0 \times 10 = 0$	0
1	$1 \times 9 = 9$	9π
2	$2 \times 8 = 16$	16π
3	$3 \times 7 = 21$	21π
	etc.	

The graph is shown in fig. 13.

Through the vertex G draw the straight line EGF parallel to the axis of x.

Draw the ordinate 5G, which represents the section through the centre of the sphere.

It is readily seen that the areas EGA and GFB are each one-third of half the rectangle AEFB, or one-sixth of the whole rectangle.

It follows that the area enclosed by AGB and the axis of x (AB) is two-thirds of the rectangle AEFB.

The rectangle AEFB represents the volume of the cylinder whose length and diameter each equal the diameter of the sphere. This cylinder is called the circum-cylinder of the sphere, and its volume is $\pi R^2 \times 2R = 2\pi R^3$.

$\therefore$ Vol. of sphere $= \frac{2}{3} \times 2\pi R^3 = \frac{4}{3}\pi R^3$.

For R = 5 cm., vol. $= \frac{4}{3}\pi(5)^3 = \frac{500}{3}\pi$ c.c.

Regarded from the line EF, the corner pieces EGA and FGB may be taken to represent two cones shown in the figure, each having altitude R and radius of base R. Their combined volume is $\frac{2}{3}\pi R^3$, each being $\frac{1}{3}\pi R^3$.

The areas of the graphs show that

circum-cylinder $(2\pi R^3)$ = **sphere** $(\frac{4}{3}\pi R^3)$ + **cones** $(\frac{2}{3}\pi R^3)$.

This relation is of importance because it holds also for parts cut off all three volumes.

This follows because the area of section of the left-hand cone (fig. 13) at distance x from A is $\pi(R - x)^2$, which is equal to $\pi R^2 - \pi x(2R - x)$, i.e. to KH, since $AE = \pi R^2$.

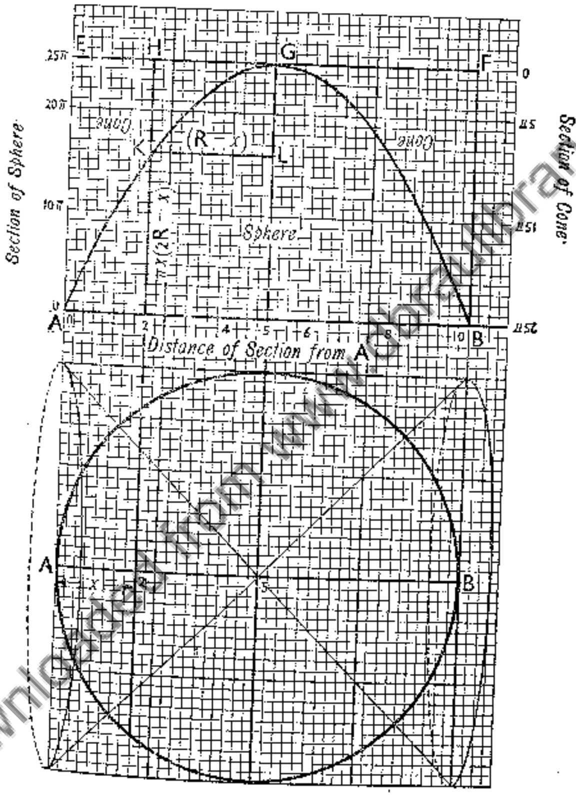


Fig. 13

E.g. the volume of the circum-cylinder from A to 2 = vol. of cap of sphere from A to 2 + vol. of part of cone from A to 2.

The relation enables you to find the volume of a cap of a sphere.

Note that the radius of any section of either of the cones is equal to the distance of the section from the apex, that is from the centre of the sphere.

Example.—Find the volume of a cap of thickness or altitude 2 cm. cut from a sphere of radius 5 cm.

Part of circum-cylinder $= \pi \times 5^2 \times 2 = 50\pi$ c.c.

Part of cone (frustum) = full cone − remaining cone

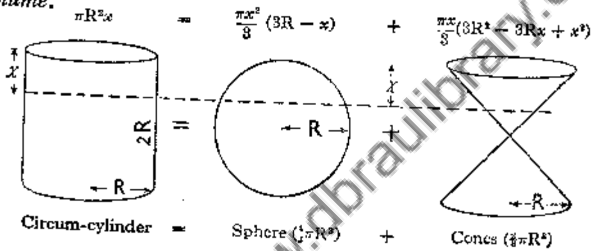
$$= \tfrac{1}{3}\pi \times 5^3 - \tfrac{1}{3}\pi \times 3^3$$

$$= \tfrac{98}{3}\pi = 32\tfrac{2}{3}\pi \text{ c.c.}$$

Vol. of cap = part of circum-cylinder − frustum of cone

$$= 50\pi - 32\tfrac{2}{3}\pi = 17\tfrac{1}{3}\pi \text{ c.c.}$$

Volume.



Area of Curved Surface.

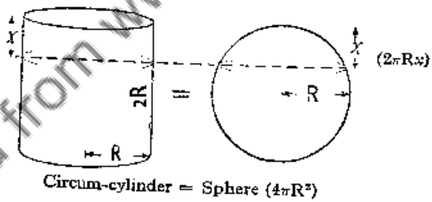


Fig. 14

The general formula for the volume of a cap of thickness x cut from a sphere of radius R can be established from the foregoing relation.

Part of circum-cylinder $= \pi R^2 x$.

Frustum of cone $= \tfrac{1}{3}\pi R^3 - \tfrac{1}{3}\pi(R - x)^3$

$$= \tfrac{1}{3}\pi(3R^2x - 3Rx^2 + x^3)$$

$$= \pi(R^2x - Rx^2 + \tfrac{1}{3}x^3).$$

Vol. of cap $= \pi R^2 x - \pi(R^2x - Rx^2 + \tfrac{1}{3}x^3)$

$$= \pi x^2(R - \tfrac{1}{3}x) \text{ or } \tfrac{1}{3}\pi x^2(3R - x).$$

The relations are illustrated in fig. 14.

9. Area of the Curved Surface of a Spherical Cap.

Compare the formula for the area of a triangle, viz. half the product of the base and the altitude, with that for the area of a sector of a circle, viz. half the product of the arc and the radius. If the arc of the sector is called the base, and the radius the altitude, the two formulæ become the same.

A similar comparison can be made between the rules for determining the volume of a cone having a plane base and a cone having a spherical base, the centre of which is the apex of the cone.

In both cases, the volume is one-third the product of the area of the base and the altitude. From this rule, the area of the curved surface of a spherical cap can be determined.

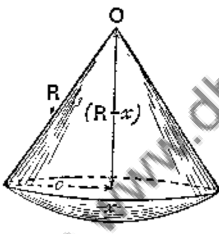


Fig. 15

Referring to fig. 15,

Vol. of "spherical" cone = Vol. of cap + vol. of "plane" cone

$$= \tfrac{1}{3}\pi x^2(3R - x) + \tfrac{1}{3}\pi c^2(R - x)$$

(substituting $x(2R - x)$ for c^2)

$$= \tfrac{1}{3}\pi x^2(3R - x) + \tfrac{1}{3}\pi x(2R - x)(R - x)$$
$$= \tfrac{1}{3}\pi x\{x(3R - x) + (2R - x)(R - x)\}$$
$$= \tfrac{1}{3}\pi x(3Rx - x^2 + 2R^2 - 3Rx + x^2)$$
$$= \tfrac{2}{3}\pi R^2 x.$$

Now $\dfrac{\text{area of base} \times \text{altitude}}{3}$ = vol. of "spherical" cone;

$$\therefore \text{ area of base} = \frac{3 \times \text{vol. of "spherical" cone}}{\text{altitude}}$$
$$= \frac{3 \times \frac{2}{3}\pi R^2 x}{R}$$
$$= 2\pi R x.$$

In the case of the hemisphere, $x = R$, and the formula becomes $2\pi R^2$. For the surface of the whole sphere, the formula is, of course, $4\pi R^2$; that is, four times the area of the circle of the same radius, or $2\pi R \times 2R$ = circumference × diameter.

Note.—The above formulæ for volumes and areas connected with a sphere can be proved simply and rigorously with the help of the Calculus. (See p. 379.)

EXAMPLE.—To find the area of the earth's surface within the horizon of an observer at a height h.

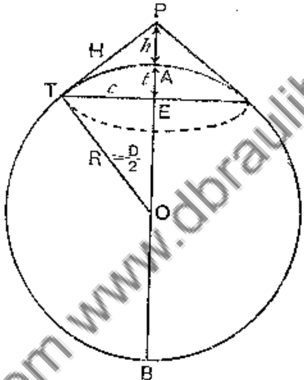


Fig. 16

Referring to fig. 16, by Geometry,

$$H^2 = h(h + D),\text{ where D is the earth's diameter.}$$

From the right-angled triangle PET,

$$c^2 + (h + t)^2 = h(h + D),$$

$$t(D - t) + (h + t)^2 = h(h + D),$$

$$Dt - t^2 + h^2 + 2ht + t^2 = h^2 + Dh,$$

from which

$$t = \frac{Dh}{D + 2h}.$$

If, as is usual, h is small compared with D, this equation reduces to

$$t = h.$$

The area of the surface of the spherical cap bounded by the circle through T is πDt.

Hence

$$\text{Area within the observer's horizon} = \pi Dt$$
$$= \pi Dh \text{ (approx.).}$$

Note.—When h is small compared with D, the arc AT is approximately equal to the tangent PT. Hence, by Chapter XIX, p. 217, the range of vision (AT) is $\sqrt{Dh}$.

EXAMPLE.—Find the area of the earth's surface within the horizon of an observer in an aeroplane $1\frac{1}{2}$ miles above the earth.

$$\text{Area} = \pi Dh = 3{\cdot}14 \times 8000 \times 1\tfrac{1}{2} \text{ sq. miles}$$
$$= 37{,}700 \text{ sq. miles (approx.).}$$

The range of vision in this case is

$$\sqrt{8000 \times 1\tfrac{1}{2}} = \sqrt{12000}$$
$$= 110 \text{ miles (approx.).}$$

EXERCISE XXII (B)

1. Referring to fig. 13, find the volume of the zone of the sphere between the sections at $x = 2$ and $x = 5$.
2. From the graph relating to the cone (p. 244), find the area of section, and then calculate the radius of a cylinder of the same altitude and volume as the cone.
3. Find the volume of the frustum of a cone the dimensions being height 10 cm., radius of top 8 cm., radius of bottom 12 cm.
4. Find the fraction left when a cone of half the altitude of a full cone is cut off.
5. On the figures showing the graphs of sections of the sphere and a cone, construct rectangles, the areas of which are equal to the areas representing the volumes of these solids.

   The altitude of each rectangle represents the mean section of the solid.

   Find, in each case, the position at which the actual section of the solid is equal to the mean section.
6. Take a spherical flask, measure its diameter, length and width of neck, etc., and calculate its volume. Verify your result by filling it with water and measuring the quantity.

7. Procure a conical flask, determine its dimensions and calculate its volume. Verify as in the last case.

8. Show that in volume, circum-cylinder of sphere : sphere : cones = 3 : 2 : 1.

9. Find the curved surface of the cap of altitude 3 in., cut from a hemisphere of radius 8 in. What is the area of the curved surface of the remaining zone?

10. Establish a formula for the curved surface of the zone of a sphere.

11. Calculate the area of the Arctic cap of the earth, and of one of the Temperate zones.

12. In testing the efficiency of an electric motor, the following numbers were found for the current and the efficiency:

Current (amps.)	0	2	5	10	12·5	15	20
Efficiency (%)	0	24·8	53·6	80·4	83·75	80·4	53·6

Plot the numbers, and find an equation connecting efficiency and current.

13. The power lost in a motor depends upon the speed. The following numbers were obtained in an experiment:

Speed (revs. per min.)	0	100	200	300	400	500	600	800	1000
Watts lost - -	0	20	48	75	110	155	200	310	430

Find the law.

14. The following numbers give the velocity of a falling body after it has fallen through the given distances. Find the law connecting the two quantities.

Distance (ft.), d -	0	2	4	9	16	25	36	64	100
Velocity (ft. per sec.), v	0	11·3	16	24	32	40	48	64	80

15. A spherical vessel, internal diameter 20 in., contains liquid to a depth of 16 in. Find the number of gallons.

16. The following rises in temperature were obtained after the currents shown were passed for the same time through a coil of wire placed in a quantity of water. Find the law.

Current (C)	0	2	3	4	6	8
Rise in temperature (θ)	0	6	$13\frac{1}{2}$	24	54	96

17. The following numbers give the distance through which a body falls from rest in various intervals of time:

Time (sec.), t	0	1	2	3	4
Distance (ft.), d	0	16	64	144	256

Plot these numbers, and from the graph deduce the law connecting d and t.

18. The table shows the available power from an electric generator when supplying the various currents given:

Power (watts)	72	128	192	168	128
Current (amps.)	2	4	8	14	16

Find the law connecting power and current, and the value of the current for which the power is a maximum.

19. A cone 10 cm. high floats, apex downwards, in water, and its apex is at such a depth that half its volume is beneath the surface. Find how far the apex is below the surface.

20. A hollow tin cone (diameter of base 12 in., altitude 12 in.), when placed, apex downwards, in water, floats with its apex 8 in. below the surface.

How much farther would it sink if water were poured into the tin cone to a height of 6 in.?

(Remember that a floating body displaces a volume of liquid whose weight is equal to that of the body.)

10. The calibration of a cylindrical petrol tank with its length horizontal makes an excellent exercise. The problem is to express depths of liquid in gallons.

Take a tank of diameter 12 in. and length 36 in. and find the volumes of liquid for depths increasing by an inch.

The volume of liquid for the depth DC (fig. 17) is the area of the segment ACB multiplied by the length of the tank.

The segment ACB is the difference between the sector OACB

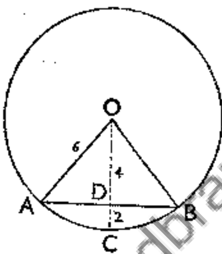


Fig. 17

and the triangle OAB. The angle of the sector is twice the angle whose cosine is $\frac{OD}{OA}$.

For the depth 2 in. the calculation is as follows:

$\text{Cos}\,\angle AOD = \frac{4}{6} = \cdot 6$. From tables, $\angle AOD = 48°$.

$\therefore \angle AOB = 96°$.

Area of sector OACB $= \frac{96}{360} \times \pi 6^2 = 30{\cdot}2$ sq. in.

$AD = 6 \sin 48° = 6 \times \cdot 7431 = 4{\cdot}46$ in.

Area of $\triangle AOB = 4{\cdot}46 \times 4 = 17{\cdot}8$ sq. in.

$\therefore$ Area of segment ACB $= 30{\cdot}2 - 17{\cdot}8 = 12{\cdot}4$ sq. in.

Vol. of liquid $= 12{\cdot}4 \times 36$ c. in.

$$= \frac{12{\cdot}4 \times 36}{1728} \times 6\tfrac{1}{4} \text{ gall.}$$

$$= 1{\cdot}6 \text{ gall.}$$

Continue the calculations to depth 6 in. From the results the gallons for depths 6 to 12 in. are easily obtained. Then graph gallons against depth and from the graph read off the depths for integral gallons, 1, 2, 3, etc. Calibrate a dipstick in gallons from the results.

11. Surfaces and Volumes of Revolution. Theorems of Pappus.*

I. If a line of a plane revolve about an axis in the plane, the area of the surface generated is equal to the product of the length of the line and the length of the path described by its centre of gravity.

II. If a plane area revolve about an axis in its plane, the volume generated is equal to the product of the area and the length of the path described by its centre of gravity.

(The axis must not intersect the generating line or area.)

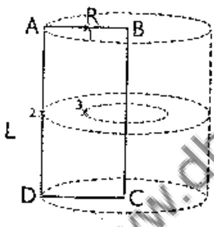


Fig. 18

The theorems will be illustrated by the revolution of a rectangle ABCD about one side (BC) as axis (fig. 18).

By a complete revolution:

(i) AB generates a circle.
(ii) AD generates the curved surface of a cylinder.
(iii) Area ABCD generates the volume of the cylinder.

(i) Calling AB, R, the C.G. of AB (1) is distance $\frac{R}{2}$ from the axis, and the length of its path is $2\pi\frac{R}{2} = \pi R$. By theorem I, area generated by AB $= R \times \pi R = \pi R^2$.

(ii) The C.G. of AD (2) is at its middle point and this is distance R from the axis. The path described is $2\pi R$. If the length of AD is L, then the area generated is, by theorem I,

$$L \times 2\pi R = 2\pi RL.$$

* The Theorems of Pappus (about A.D. 300) are sometimes attributed to Guldinus, who rediscovered or revived them about 1640.

(iii) The C.G. of rectangle ABCD (3) is distance $\frac{R}{2}$ from the axis. The path described is $2\pi\frac{R}{2} = \pi R$. The area of ABCD is RL and by theorem II,

$$\text{Volume generated} = RL \times \pi R = \pi R^2 L,$$
$$\text{i.e. area of base} \times \text{height.}$$

These formulæ agree with those already established.

Application of the Theorems.

1. *Anchor-ring or Tore.*

The ring is generated by the revolution of a circle about an axis in the same plane.

Let r be the radius of the circle and R the distance of its centre from the axis (fig. 19).

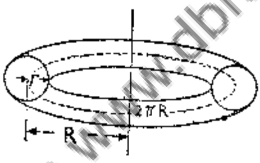


Fig. 19

The centre of gravity of both the circumference and the area of the circle is its centre, and the path described in a revolution round the axis is therefore $2\pi R$.

By theorem I, surface generated by the circumference of the circle

$$= 2\pi r \times 2\pi R = 4\pi^2 rR.$$

By theorem II, volume generated by the area of the circle

$$= \pi r^2 \times 2\pi R = 2\pi^2 r^2 R.$$

2. The theorems can be used to calculate the position of certain centres of gravity, by means of known formulæ of surface and volume.

For example, the arc of a semicircle rotating about the diameter as axis generates the surface of a sphere, and the area of the semicircle the volume of the sphere. From the known formulæ the position of the C.G. of a semicircular arc and the C.G. of a semicircle can be found.

By symmetry, the C.G.s are along the middle radius.

Let x = the distance of the C.G. from the diameter as axis (fig. 20).

For the semicircular arc, by theorem I,

$$\text{Arc} \times \text{path of C.G.} = \text{Surface of sphere.}$$

$$\pi R \times 2\pi x = 4\pi R^2.$$

$$\therefore\ x = \frac{2R}{\pi} = \frac{7}{11}R \text{ (approx.)}.$$

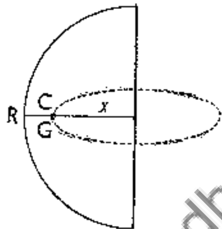


Fig. 20

For the area of the semicircle, by theorem II,

$$\text{Area} \times \text{path of C.G.} = \text{Volume of sphere.}$$

$$\tfrac{1}{2}\pi R^2 \times 2\pi x = \tfrac{4}{3}\pi R^3.$$

$$\therefore\ x = \frac{4R}{3\pi} = \frac{14}{33}R.$$

Exercise XXII (c)

1. Find the volume and surface of a ring the outside diameter of which is 24 in. and the inside diameter 12 in.
2. Show that the C.G. of the complete boundary of a semicircle is $\frac{7}{18}$R along the middle radius from the diameter.
3. Calculate the surface and volume generated by the revolution of a semicircle of 6 in. radius about an axis in the same plane, parallel to and 18 in. from the diameter, measured away from the semicircle.
4. Find the surface and volume of a ring of square section, the inner and outer diameters being 24 and 30 in. respectively.

CHAPTER XXIII

TRIGONOMETRY, APPLICATIONS TO MECHANICS, ETC.

1. Trigonometry.

Trigonometrical Ratios and their Relations.

It is important to note how powers of the trigonometrical ratios are written. E.g. the square of sin A is written $\sin^2 A$.

Again,
$$\sin 45^\circ = \frac{1}{\sqrt{2}};$$
$$\therefore \sin^2 45^\circ = \frac{1}{2}.$$

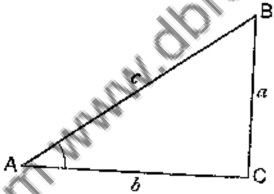


Fig. 1

Referring to the right-angled triangle ABC (fig. 1),
$$a^2 + b^2 = c^2.$$

Divide all through by c^2; then
$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1,$$
or
$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1;$$
$$\therefore \mathbf{\sin^2 A + \cos^2 A = 1.}$$

EXERCISE.—(1) Arrange this for finding:

(i) sin A in terms of cos A.

(ii) cos A in terms of sin A.

(2) Choose any angle, and from the values given in the tables verify this relation.

2. Names of the Reciprocals of the three Trigonometrical Functions given.

The reciprocal of the sine ratio is called the *cosecant*—written shortly *cosec*. Thus, referring to fig. 1, cosec A is $\frac{c}{a}$.

The reciprocal of the cosine ratio is called the *secant*—written shortly *sec*. Thus sec A is $\frac{c}{b}$.

The reciprocal of the tangent ratio is called the *cotangent*—written shortly *cot*. Thus cot A is $\frac{b}{a}$.

Summarize these statements in the form:

$$\cot A = \frac{1}{\tan A}, \quad \sec A = \frac{1}{\cos A}, \quad \operatorname{cosec} A = \frac{1}{\sin A}.$$

3. Starting with the relation

$$a^2 + b^2 = c^2,$$

and dividing in the first instance by a^2, and in the second by b^2, find relations corresponding to that in **1**. Your results should be:

(i) $\operatorname{cosec}^2 A - \cot^2 A = 1$. (ii) $\sec^2 A - \tan^2 A = 1$.

All these relations, and the method of establishing them, should be remembered.

Exercise XXIII (a)

1. Making use of the reciprocal functions, find cosec 30°, sec 30°, cot 30°, and these functions of 45°, 60° and 90°.
2. Give the reason for the following equalities:
$$\sin^2 A + \cos^2 A = \operatorname{cosec}^2 A - \cot^2 A = \sec^2 A - \tan^2 A.$$
3. If $t = \frac{V^2}{g}\cos^2 A$, find V when t is 5, A 30° and g 32.
4. Express $\cos^2 A$ in terms of $\sec^2 A$, then convert $\sec^2 A$ into $\tan^2 A$, and so obtain $\cos^2 A$ in terms of $\tan^2 A$; continuing, obtain $\cos^2 A$ in terms of $\operatorname{cosec}^2 A$.

   Beginning again with $\cos^2 A$, obtain $\cos^2 A$ in terms of $\sin^2 A$, and from this, in terms of $\operatorname{cosec}^2 A$.

   See that the two results agree.
5. Find sin A in terms of sec A.

6. Find $\cos A$ when $\tan A = \frac{2}{3}$, and $\cos A$ when $\cot A = \frac{5}{4}$.

7. Show that

$$\text{(i)}\quad \sin^2 A + \tan^2 A = \sec^2 A - \cos^2 A.$$
$$\text{(ii)}\quad \sin^2 A(1 + \cos^2 A) = 1 - \cos^4 A.$$

8. In fig. 1, the angle B is equal to $(90° - A)$; hence find the trigonometrical functions of $(90° - A)$ in terms of functions of A. E.g. $\operatorname{cosec} B = \operatorname{cosec}(90° - A) = \frac{c}{b} = \sec A$. Thus:

$$\begin{aligned} \cos(90° - A) &= \sin A; & \sin(90° - A) &= \cos A; \\ \cot(90° - A) &= \tan A; & \tan(90° - A) &= \cot A; \\ \operatorname{cosec}(90° - A) &= \sec A; & \sec(90° - A) &= \operatorname{cosec} A. \end{aligned}$$

9. Find $\tan A$ in terms of (i) $\sin A$, (ii) $\cos A$.

10. Plot the values of the six trigonometrical ratios for various values of x, where x represents the angle from 0° to 90°.

4. The following applications to Mechanics are important:

(1) If OX and OY represent forces acting at a point O (fig. 2), then OR, the diagonal of the parallelogram OXRY, represents * their resultant. (Lines like OX, OY and OR are called Vectors.)

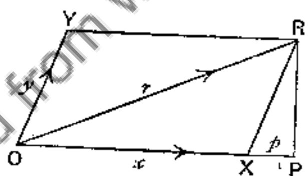


Fig. 2

From the forces OX and OY, and the angle between their directions, the value of OR can be calculated.

Draw RP perpendicular to OX, or to OX produced.

XP is called the projection of XR on OX.

Observe that $\angle RXP = \angle YOX$.

Now
$$\frac{XP}{XR} = \cos\angle RXP$$
$$= \cos\angle YOX;$$
$$\therefore\ XP = XR \times \cos\angle YOX. \qquad \text{(i)}$$

* Discovery attributed to Aristotle (384–322 B.C.).

Let the forces be x and y, the angle between their directions A, the projection XP, p, and the resultant of x and y, r.

Then, by Geometry,

$$r^2 = x^2 + y^2 + 2xp$$
$$= x^2 + y^2 + 2xy \cos A^* \text{ (from (i), and since XR = OY)},$$

i.e. $r = \sqrt{x^2 + y^2 + 2xy \cos A}$. . . . . (ii)

The angle ROX, which OR makes with OX, is determined as follows:

$$OP = OX + XP = x + y \cos A$$

and $$\cos \angle ROX = \frac{OP}{OR} = \frac{x + y \cos A}{r},$$

i.e. the angle ROX is such that its cosine is equal to $\frac{x + y \cos A}{r}$.

The method of writing such an angle is $\cos^{-1}\left(\frac{x + y \cos A}{r}\right)$.

Thus, $\cos^{-1}\frac{1}{2}$ means, the angle whose cosine is $\frac{1}{2}$. What is this angle?

Example.—Find the resultant of forces of 10 lb. and 6 lb. acting at an angle of 50°, at the same point.

$$r = \sqrt{(6)^2 + (10)^2 + 2 \times 6 \times 10 \times \cos 50^\circ}$$
$$= \sqrt{136 + 120 \times \cdot6428}$$
$$= \sqrt{136 + 77\cdot14}$$
$$= \sqrt{213\cdot14}$$
$$= 14\cdot6 \text{ lb.}$$

The direction of r is such that the cosine of the angle which it makes with the direction of the force 6 lb. is equal to

$$\frac{6 + 10 \cos 50}{14\cdot6} = \frac{6 + 10 \times \cdot6428}{14\cdot6}$$
$$= \frac{12\cdot428}{14\cdot6}$$
$$= \cdot8511.$$

From tables $\cos^{-1} \cdot8511 = 32^\circ$ (approx.),

i.e. the resultant makes an angle of 32° with the direction of the force 6 lb.

* When A is obtuse, cos A is negative.

(2) The vector principle is applicable to velocity.

When, for example, a marble is rolled across the floor of a moving railway carriage, the velocity of the marble regarded from the ground is the resultant of the velocity across the floor and the velocity of the floor. The velocity of an aeroplane is practically the resultant of the velocity it would have in calm air and the velocity of the wind.

Imagine that an aeroplane travelling N.E. at 80 miles per hour in a calm area, suddenly enters an area in which a W. wind is blowing at 30 miles per hour. Find the resultant direction and speed. (Remember, a west wind blows towards the east.)

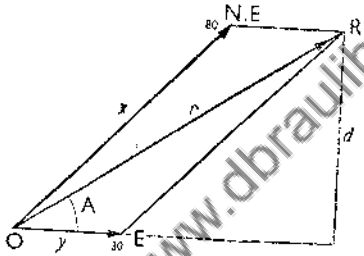


Fig. 3

From fig. 3:

$$\begin{aligned} r^2 &= x^2 + y^2 + 2xy \cos 45° \\ &= 6400 + 900 + 4800 \times \cdot 707 \\ &= 7300 + 3393\cdot 6 \\ &= 10{,}693\cdot 6; \\ \therefore\ r &= 103\cdot 4 \text{ miles an hour.} \end{aligned}$$

You are left to find the angle A either by the method on p. 261 or by finding d and applying $\sin A = \frac{d}{r}$, or from $\frac{\sin A}{x} = \frac{\sin 135°}{r}$.

Note.—When x and y are determined from r, r is said to be *resolved* into its *components*. If the directions of x and y are at right angles to one another, and A is the angle between r and x, then

$$x = r \cos A, \quad \text{and} \quad y = r \sin A.$$

Draw a figure and verify these. Show also that $x^2 + y^2 = r^2$.

5. Relative Velocity.

It was mentioned as early as in Chapter IV that in finding differences we may regard the zero as being moved to the number from which we have to view the difference. Relative velocities are determined in the same way.

EXAMPLE i.—Two trains, A and B, are running in the same direction on parallel lines. If A goes at 60 miles an hour and B at 20 miles an hour, find the relative velocity of A to B, and of B to A. In other words, find at what velocity A appears to be going to a person in B, and at what velocity B appears to be going to a person in A.

(1) If the velocity of B is regarded as the zero, then, considered from this zero, the velocity of A is (60 − 20), i.e. 40 miles per hour, and is in the same direction as that of A.

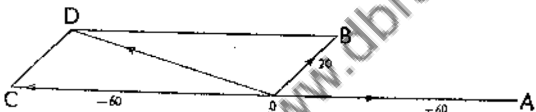


Fig. 4

(2) If the velocity of A is regarded as the zero, then relative to this zero the velocity of B is (20 − 60), i.e. −40 miles per hour. The negative sign shows that, relative to A, the velocity of B is reversed. To a person in train A, train B appears to be going backwards at a rate of 40 miles per hour.

These statements are borne out by experience.

EXAMPLE ii.—Two trains, A and B, pass a junction, A travelling east at 60 miles per hour and B north-east at 20 miles per hour. Find their relative velocities.

The relative velocity of B, regarded from A, is found as follows: OA and OB are drawn to represent in magnitude and direction the actual velocities of the trains.

The velocity of train A is reduced to the zero by impressing upon it a velocity equal but opposite to that represented by OA. The same velocity is impressed upon B, and the resultant found.

In fig. 4, OD is the resultant, OC being the impressed velocity. The resultant OD represents the velocity of B relative to A.

This is seen to be correct, for when train A reaches a point corresponding to A in the figure, train B will have a position corresponding to point B, and therefore a person in A must look in the direction AB, which is parallel to OD, to see train B.

The velocity of A with respect to B is found by drawing OB in the reverse direction, and impressing this reversed velocity on A and finding the resultant. Determine this.

6. Triangle of Forces or Velocities.*

The triangle OXR (fig. 2) has its sides parallel to the forces x, y and r, and may be used instead of the parallelogram. A force equal in magnitude but exactly opposite in direction to the resultant r would balance it and therefore also x and y, and thus the three forces x, y and the force equal but opposite to r would be in equilibrium.

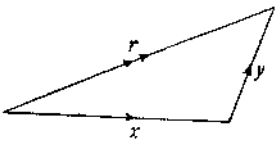


Fig. 2*a*

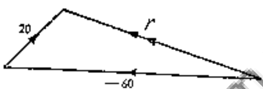


Fig. 4*a*

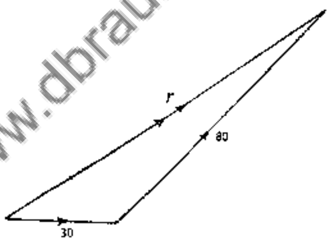


Fig. 3*a*

Note carefully that the directions then run round the triangle, that is, O to X, X to R, R to O respectively.

It follows also that any one force reversed is the resultant of the other two.

Figs. 2(*a*), 3(*a*), and 4(*a*) show the triangles corresponding to figs. 2, 3, and 4.

7. Projection of an Area.

Consider a rectangle ABCD, the plane of which makes an angle with the plane of the page of this book (fig. 5).

(This angle is measured between straight lines drawn one in each plane, from a point on the line of intersection of the planes, and at right angles to it (see p. 21)).

In the figure, AD is the line of intersection, and if A*b* and AB

* Attributed to Stevinus, A.D. 1548–1620 (nineteen centuries after Aristotle!).

are at right angles to AD, $\angle BAb$ is the angle between the planes.

If Bb and Cc are drawn at right angles to the plane of the page, then $AbcD$ is called the projection of ABCD on the plane of the page.

When ABCD is a rectangle, $AbcD$ also is a rectangle.

Let angle BAb be denoted by A; then $Ab = AB \cos A$.

$$\text{Area of } AbcD = AD \times Ab$$
$$= AD \times AB \cos A$$
$$= \text{area of ABCD} \times \cos A,$$

i.e. the area of the projection of the area ABCD on the plane of the page is equal to the area of ABCD multiplied by the cosine of the angle between the planes.

It follows also that area of ABCD $= \dfrac{AbcD}{\cos A}$.

This relation is quite general. It is true for plane areas of all shapes.

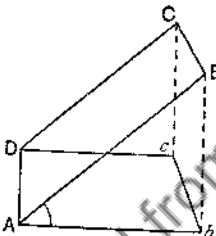


Fig. 5

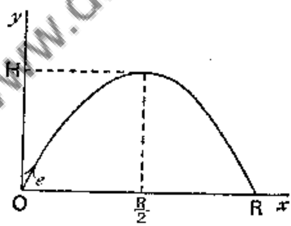


Fig. 6

8. The Path of a Projectile.

The path or trajectory of a projectile is approximately a parabola, and may be taken as such for theoretical considerations.

If the point of projection is the origin, then the equation of the path is of the form

$$y = ax^2 + bx \quad \text{(since } c \text{ is 0).} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

See fig. 6.

The coefficient of x^2 is of course negative.

If H is the greatest altitude attained, and R the horizontal range of the projectile, we know from Chapter XX, 2, that

$$R = -\frac{b}{a} \quad \text{and} \quad H = -\frac{b^2}{4a}.$$

From these relations it readily follows that

$$b = \frac{4H}{R} \quad \text{and} \quad a = -\frac{4H}{R^2}.$$

Hence, in terms of the horizontal range and the maximum altitude, the equation to the path is

$$y = -\frac{4H}{R^2}x^2 + \frac{4H}{R}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Now H and R depend upon the angle of elevation of the gun and on the velocity of projection.

Let V = the velocity of projection in, say, feet per second, and e = the angle of elevation of the gun (more correctly, the "quadrant" angle). Then, resolving the velocity, vertically and horizontally (see Note, section **4**, p. 262),

Vertical velocity = V sin e, and horizontal velocity = V cos e.

If the vertical velocity is subject to an acceleration of $-g$ during ascent and g during descent (g is approximately 32 ft. per sec. per sec.), then the time to reach the greatest altitude is $\frac{V \sin e}{g}$ and the time of flight double this, namely $\frac{2V \sin e}{g}$.

Now the horizontal velocity is constant, and therefore:

The horizontal range, R = horizontal velocity × time

$$= V \cos e \times \frac{2V \sin e}{g}$$

$$= \frac{2V^2}{g} \sin e \cos e.$$

Hence

$$R = \frac{2V^2}{g} \sin e \cos e. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Again, from Ex. XXII (A), 1,

$$H = \tfrac{1}{2}gt^2;$$

and since

$$t = \frac{V \sin e}{g},$$

therefore

$$H = \frac{V^2 \sin^2 e}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Equation (ii) now becomes

$$y = \frac{\dfrac{-4V^2 \sin^2 e}{2g}}{\dfrac{4V^4 \sin^2 e \cos^2 e}{g^2}} x^2 + \frac{\dfrac{4V^2 \sin^2 e}{2g}}{\dfrac{2V^2 \sin e \cos e}{g}} x$$

$$= -\frac{g}{2V^2 \cos^2 e} x^2 + \frac{\sin e}{\cos e} x,$$

or, since $\frac{1}{\cos^2 e} = \sec^2 e$ and $\frac{\sin e}{\cos e} = \tan e$,

$$y = -\frac{g}{2V^2} x^2 \sec^2 e + x \tan e.* \quad \quad \text{(v)}$$

By means of this equation, the altitude (y) at any point of the flight can be determined.

EXERCISE XXIII (B)

1. From the tables, find the cosecant, secant and cotangent of: 20°, 50°, 75°, 85°.
2. Resolve a force of 100 lb. in directions making 60° and 30° on different sides of the direction of the force.
3. By means of a rope, a horse exerts a force of 200 lb. upon a railway truck. If the rope makes an angle of 35° with the rails, calculate the force urging the truck along the rails when both rope and rails are horizontal.
4. A train is going E. at the rate of 80 ft. per sec. A rifle, held at right angles to the train, is discharged by a passenger, and it is found that the bullet follows a horizontal course 81° N. of E. Find the velocity of the bullet.
5. If the area ABCD (fig. 5) is 20 sq. in., find the area of its projection on a plane making 50° with it.
6. A cylinder, 10 in. diameter, is cut by a plane making 35° with its axis. Calculate the area of the section. Determine also the axes of the section, and check your first result by calculating the area from the lengths of the axes.

   (Area = π times the product of the semi-axes.)

* A simpler proof of this equation is given in Chapter XXVIII.

7. Find the resultant of the following velocities:
   (i) 75 m.p.h. N. and 30 m.p.h. N.
   (ii) 75 m.p.h. N. and 30 m.p.h. S.
   (iii) 75 m.p.h. N. and 30 m.p.h. 60° clockwise from N.
   (iv) 75 m.p.h. 30° clockwise from N. and 30 m.p.h. 150° clockwise from N.

8. Find the horizontal range of a gun having a muzzle velocity of 2000 ft. per second, the quadrant angle being 30°. What altitude will the projectile attain?

9. At what angle must a gun be set to hit an aeroplane 6000 ft. high at a horizontal distance of 1000 yd., the muzzle velocity being 2000 ft. per second?

10. An aviator is flying at 200 miles per hour in a direction 25° N. of E., and a west wind is blowing 60 miles per hour. What is the direction of the wind relative to the aviator?

11. An aeroplane is flying horizontally at 200 m.p.h. at a height of 10,000 ft. Find at what distance from a ground target a bomb should be released to score a hit. (First find the time to fall 10,000 ft. vertically.)

12. Assuming the trajectory of a projectile to be a parabola, determine its equation when the maximum altitude is 2 miles and the horizontal range 12 miles.

13. Determine the equation to the parabolic trajectory which has a maximum altitude of 12,000 ft. and a horizontal range of 15,000 ft.

14. Arrange equation (v) (p. 267) in a convenient form for calculating V.
    Calculate V when $e = 15°$, $y = 1$ mile, and $x = 4$ miles. Under what conditions would V be imaginary?

15. If t is the full time of flight of a projectile, show that
$$V^2 = \tfrac{1}{4}g^2t^2 + R^2/t^2.$$

9. The Trigonometrical Functions of Angles of any Magnitude.

So far the trigonometrical ratios of angles between 0° and 90° only have been considered. It is now necessary to consider angles of any magnitude.

Imagine a straight line OP (fig. 7) to rotate about the point O in, say, the counter-clockwise direction. Let OX be the initial direction of OP.

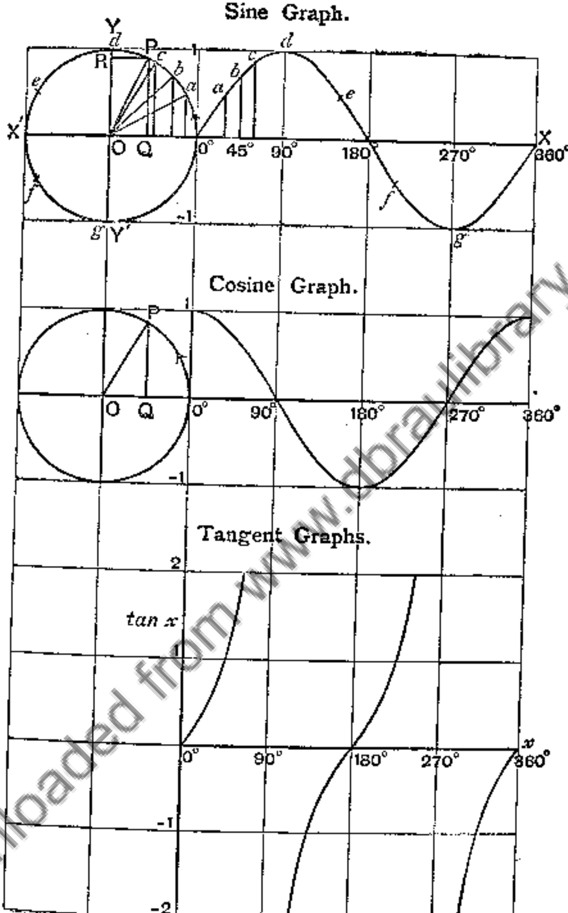


Fig. 7

When the line has reached the position OP shown, it has rotated through the angle XOP.

Draw PQ at right angles to OX; then

$$\sin \angle XOP = \frac{PQ}{OP} \quad \text{and} \quad \cos \angle XOP = \frac{OQ}{OP}.$$

If PR is drawn at right angles to OY, then PQ = OR and

$$\sin\angle XOP = \frac{OR}{OP}.$$

Now, OR is the projection of the rotating line OP on the vertical line OY, and OQ the projection of OP on the horizontal line OX.

A more general definition of each ratio is as follows:

$$\text{sine}\ \angle XOP = \frac{\text{Projection of OP on the vertical axis}}{\text{Rotating line OP}},$$

$$\text{cosine}\ \angle XOP = \frac{\text{Projection of OP on the horizontal axis}}{\text{Rotating line OP}},$$

$$\text{tangent}\ \angle XOP = \frac{\text{Projection of OP on the vertical axis}}{\text{Projection of OP on the horizontal axis}}.$$

These definitions can be applied to angles greater than 90°. *The projections are positive if measured in the direction* OX *or* OY, *and negative if in the direction* OX′ *or* OY′.

Note carefully the following:

(1) When OP has the direction OX, the length of the projection on the vertical axis is 0, and that on the horizontal is equal to the length of OP.

Hence, $\sin 0° = 0$, $\cos 0° = \frac{OP}{OP} = 1$ and $\tan 0° = 0$.

(2) When OP has rotated through 90°, and has therefore the direction OY, the length of the projection on the vertical axis is equal to the length of OP, and that on the horizontal axis is 0.

Hence, $\sin 90° = 1$, $\cos 90° = 0$ and $\tan 90° = \frac{1}{0} = \infty$.

(3) When OP is in the second quadrant, i.e. when the angle XOP is obtuse, the projection of OP on the horizontal is negative. The cosine and tangent of angles between 90° and 180° are therefore negative.

(4) When OP is in the third quadrant, both projections are negative. The sine and cosine of angles between 180° and 270° are therefore negative.

(5) When OP is in the fourth quadrant, the projection on the vertical is negative.

The sine and tangent of angles between 270° and 360° are therefore negative.

The graphs of the sine, cosine and tangent of angles from 0° to 360° are shown in fig. 7. The angle is represented on the axis of x and the value of the function on the axis of y.

Examine the graph of $\tan x$ in the neighbourhood of 90° and of 270°. Observe that for an angle slightly less than 90°, $\tan x$ is positive and numerically large, and for an angle slightly greater than 90°, $\tan x$ is negative and numerically large.

It will be seen that in passing through 90° and 270°, $\tan x$ changes sign from positive to negative, and that for these values of x, $\tan x$ is infinite.

10. From fig. 7, the following relations can be verified:

$$\sin(180^\circ - x) = \sin x, \qquad \sin(180^\circ + x) = -\sin x,$$
$$\cos(180^\circ - x) = -\cos x, \qquad \cos(180^\circ + x) = -\cos x,$$
$$\tan(180^\circ - x) = -\tan x, \qquad \tan(180^\circ + x) = \tan x.$$

By means of slight additions to fig. 7, the following are also easily proved:

$$\sin(90^\circ + x) = \cos x,$$
$$\cos(90^\circ + x) = -\sin x,$$
$$\tan(90^\circ + x) = -\cot x.$$

It is seen also that:

$$\sin(90^\circ - x) = \cos x,$$
$$\cos(90^\circ - x) = \sin x,$$
$$\tan(90^\circ - x) = \cot x.$$

11. Functions of Negative Angles.

Fig. 8 shows angles +A and −A measured from the direction OX.

It is evident from the figure that the projection of OP on the vertical when in its final position after describing the angle −A is opposite in sign to the corresponding projection when in the final position after describing the angle +A. Thus, in the given figure, ON is positive and OM negative.

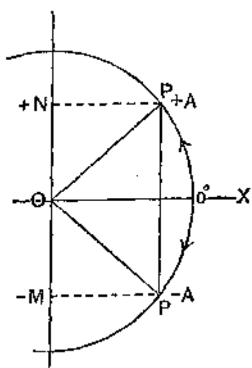


Fig. 8

On the other hand, the projections on the horizontal are exactly alike.

Hence
$$\mathbf{\sin(-A) = -\sin A,}$$
$$\mathbf{\cos(-A) = \cos A,}$$
$$\mathbf{\tan(-A) = -\tan A.}$$

12. The relation $\sin(A+B) = \sin A \cos B + \cos A \sin B$ is readily proved as follows, when A and B are together less than 180°.

Let A and B be angles of a triangle ABC (fig. 9). Then

$$C = 180° - (A+B) \text{ and } \sin C = \sin\{180° - (A+B)\} = \sin(A+B).$$

Now, $$c = a\cos B + b\cos A, \quad \text{(see Chapter XXI)} \quad . \quad \text{(i)}$$

and since
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$
$$c = \frac{a\sin C}{\sin A} \quad \text{and} \quad b = \frac{a\sin B}{\sin A}.$$

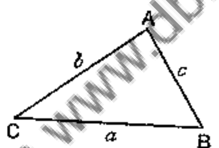


Fig. 9

Hence, substituting these values in equation (i), we have:
$$\frac{a\sin C}{\sin A} = a\cos B + \frac{a\sin B}{\sin A}\cos A,$$
from which $$\sin C = \sin A\cos B + \sin B\cos A \quad . \quad . \quad . \quad \text{(ii)}$$
and $$\therefore \mathbf{\sin(A+B) = \sin A\cos B + \sin B\cos A.} \quad . \quad . \quad . \quad \text{(iii)}$$

By similar reasoning, but beginning with
$$\cos C = \cos\{180 - (A+B)\} = -\cos(A+B),$$
then $$\cos(A+B) = -\cos C, \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$
and since $$a = b\cos C + c\cos B,$$
we have, $$-\cos C = \frac{c}{b}\cos B - \frac{a}{b}$$
$$= \frac{\sin C}{\sin B}\cdot\cos B - \frac{\sin A}{\sin B}.$$

From (ii)

$$= \frac{(\sin A \cos B + \cos A \sin B)\cos B - \sin A}{\sin B}$$

$$= \frac{\sin A \cos^2 B + \cos A \cos B \sin B - \sin A}{\sin B}$$

$$= \frac{\sin A(1 - \sin^2 B) + \cos A \cos B \sin B - \sin A}{\sin B}$$

$$= -\sin A \sin B + \cos A \cos B.$$

∴ from (iv)

$$\mathbf{\cos(A + B) = \cos A \cos B - \sin A \sin B.} \quad . \quad . \quad \text{(v)}$$

Note.—The relation (v) can be deduced from (iii) by changing A into 90° + A in (iii), and using the formulæ

$$\sin(90^\circ + x) = \cos x, \quad \cos(90^\circ + x) = -\sin x \quad \text{(section 9).}$$

Again, $\tan(A + B) = \dfrac{\sin(A + B)}{\cos(A + B)}$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing above and below by cos A cos B $\Big\}$ $= \dfrac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}.$

$$\mathbf{\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

13. The three very important formulæ for sin (A + B), cos (A + B), tan (A + B) given in the last section, were only proved for the case when A and B are positive angles less than 180°. In point of fact, they are true for all angles whatever, positive or negative, as may easily be verified for particular values of A and B, with the help of the relations of sections 9 and 10.

Now change B into −B in each of the three formulæ, using section **11**. Thus:

$$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B)$$

or

$$\mathbf{\sin(A - B) = \sin A \cos B - \cos A \sin B.} \quad . \quad . \quad . \quad \text{(vii)}$$

Similarly, $\cos(A - B) = \cos A \cos B + \sin A \sin B$, . . . (viii)

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \qquad \text{(ix)}$$

The six relations:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

are very important and should be memorized.

If B = A, then

$$\sin 2A = 2 \sin A \cos A, \qquad \text{(x)}$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A, \qquad \text{(xi)}$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}. \qquad \text{(xii)}$$

If $A = \frac{x}{2}$, then from $\cos 2A$ (xi)

$$\cos x = 2\cos^2\frac{x}{2} - 1 = 1 - 2\sin^2\frac{x}{2}, \qquad \text{(xiii)}$$

from which

$$\cos\frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}} \text{ and } \sin\frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}}, \qquad \text{(xiv and xv)}$$

and

$$\tan\frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}}. \qquad \text{(xvi)}$$

Also, from (x) and (xiii)

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}. \qquad \text{(xvii)}$$

If $A + B = x$ and $A - B = y$, then $A = \frac{1}{2}(x + y)$ and $B = \frac{1}{2}(x - y)$.

By adding and subtracting equations (iii) and (vii),

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B,$$
$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B.$$

Substituting the new values of A and B,

$$\sin x + \sin y = 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y), \quad \text{(xviii)}$$
$$\sin x - \sin y = 2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y). \quad \text{(xix)}$$

Similarly from equations (v) and (viii),

$$\cos x + \cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y), \quad \text{(xx)}$$
$$\cos x - \cos y = -2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y). \quad \text{(xxi)}$$

The student should practise deducing these relations, beginning with sin(A + B) and cos(A + B).

EXERCISE XXIII (c)

1. Examine the sine graph and trace the change in the sine of an angle as the angle increases by, say, 15° from 0° to 360°.
2. Repeat Exercise 1 with the cosine and tangent graphs.
3. Examine the graphs of Exercises 1 and 2, and make a list of all the equalities you can find.
4. Write down the sin, cos and tan of the following angles:

   120°, 135°, 150°, 180°, 210°, 225°,
   240°, 270°, 300°, 315°, 330°, 360°.
5. By how many degrees is the cosine curve in advance of the sine curve?
6. Trace the graph of $2\sin x$, and on the same axes, the graph of $\cos x$. Then add the ordinates of the two graphs together and obtain another curve. It is the graph of $2\sin x + \cos x$.
7. Trace the graph of $2\sin x - \cos x$.
8. Construct the graph of $\sin^2 x$, i.e. the square of the sine of x.
9. Draw the graph of $\log_{10}\sin x$.
10. Procure a thin pasteboard tube, and cut it across its axis at an angle, slit one part along its length, open it out flat, place it flat on squared paper, and draw a pencil line on the paper along the curved edge of the open tube. What kind of a curve does it appear to be? Verify by measurements.

Exercise XXIII (d)

1. Taking A = 45° and B = 30°, find the trig. ratios of 75° and 15° from relations (iii), (v to ix).
2. Calculate sin, cos and tan $22\frac{1}{2}°$.
3. From the values of the trigonometrical ratios of 60° and 45°, find those of 105°.
4. For what value of a will
   (i) $\sin 2a$ be a maximum? (ii) $\sin 2a = 0$?
   (iii) $\cos 2a$ be a maximum? (iv) $\cos 2a = 0$?
5. If $\sin A = x$, find $\sin 2A$, $\cos 2A$, $\sin\frac{A}{2}$, and $\cos\frac{A}{2}$ in terms of x.
6. Express $\sin^2 2x$ in terms of $\sin x$.
7. Find $\sin A$ in terms of $\sin\frac{1}{2}A$.
8. Show that

$$(\cos A + \sin A)^2 = 1 + \sin 2A.$$

9. Show that

$$\text{(i)}\ \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2},$$

$$\text{(ii)}\ 1 + \cos x = 2\cos^2\frac{x}{2},$$

and that therefore

$$\text{(iii)}\ \frac{\sin x}{1+\cos x} = \tan\frac{x}{2}.$$

10. Draw the graphs of (i) $\operatorname{cosec} x$, (ii) $\sec x$, (iii) $\cot x$, from $x = 0°$ to $x = 360°$.
11. In fig. 7 (p. 269), the diagram of the rotating radius is usually called the clock diagram of the graph.

    In Questions 6 and 7, Ex. XXIII (c), show the clock diagram of each graph, including the resultant graph.
12. On the same axes, plot the curves $\sin x$ and $\sin(x + 30°)$.

    On the same clock diagram, draw pointers to represent the rotating radii.

    Determine the resultant curve, and insert in the clock diagram the pointer corresponding to this curve. Join the free end of this pointer to the free ends of the other pointers, and see what figure is obtained (fig. 10).

13. Plot the graphs of $\sin 2x$ and $\sin x$ on the same axes.

Draw also the clock diagram as in the previous question.

Determine the resultant graph, and insert in the clock diagram the corresponding pointer.

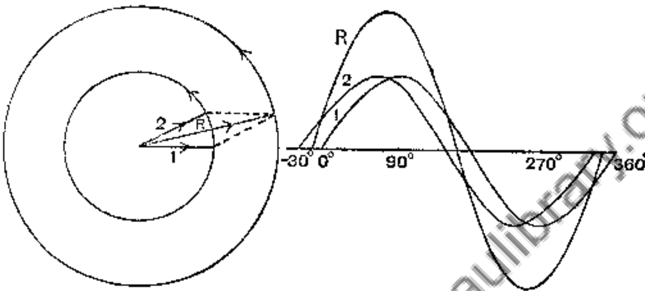


Fig. 10

14. Simplify the following:

(i) $\sin(A + B) + \sin(A - B)$. (ii) $\sin(A + B) - \sin(A - B)$.

(iii) $\cos(A + B) + \cos(A - B)$. (iv) $\cos(A + B) - \cos(A - B)$.

(v) $\tan(A + B) + \tan(A - B)$. (vi) $\tan(A + B) - \tan(A - B)$.

15. If the length of each of a train of waves is 342·1 m. and the velocity is 300 million metres per second, find the number of cycles per second. Similarly, check other wave-lengths and cycles given in the *Radio Times*.

16. (i) Beginning with the relation

$$a^2 = b^2 + c^2 - 2bc\cos A$$

and applying relations (xiv) and (xv) (p. 274) and factorizing as in **2**, p. 232, show that

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

and $$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

(ii) Find the angles of the triangle whose sides are 15, 12, and 9 cm.

CHAPTER XXIV

SPHERICAL TRIGONOMETRY

In this chapter the fundamental relations of spherical trigonometry, so essential to sailor and aviator navigators, are established, and their simpler applications considered.

1. The Spherical Triangle.

The plane sections of a sphere are circles. The largest circles are those made by sections through the centre. They have the same centre and radius as the sphere and are called GREAT CIRCLES. The circles of longitude and the equatorial circle of the earth approximate to great circles. The circles of latitude, except the equator, on the other hand, are not great circles. It is important to navigators to be able to calculate the great circle distance between two places on the earth's surface, that is, to find the length of the great circle arc between them.

The sides of a spherical triangle are arcs of great circles. This fact should be borne in mind throughout the following considerations.

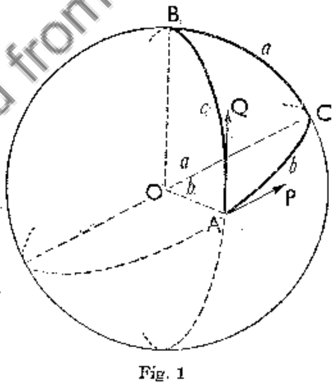


Fig. 1

In fig. 1, ABC represents such a triangle.

The lengths of the arcs are a, b, c. The angles subtended at the centre O by the arcs BC, CA, AB are proportional to these arcs, since arc BC = radius of sphere × number of radians in a.

In the trigonometrical formulæ, cos a always means the cosine of the angle BOC, and so on. If the sphere is of unit radius the arc and the angle are the same.

If tangents AP and AQ are drawn at A to arcs AC and AB respectively, the angle between these tangents is the angle between the arcs and also between the planes of the great circles of which AC and AB are arcs;

i.e. ∠PAQ is the **angle** A of spherical triangle ABC.

It should be realized that the tangents are both at right angles to the radius OA, but that the right angles are in different planes, ∠OAP being in plane OAC and ∠OAQ in plane OAB. Similar conditions hold for angles B and C.

2. *Important relations between the sides and angles of a spherical triangle can be established as follows:*

In fig. 2,* drawn in perspective, ABC is a spherical triangle, O the centre of the sphere, of unit radius. BD is ⊥ to plane AOC,

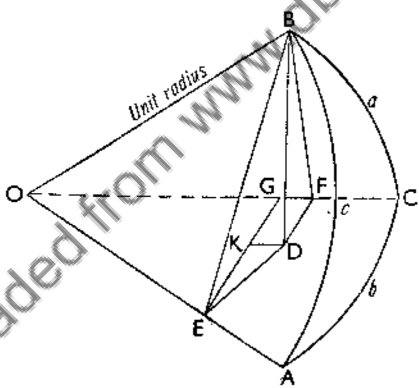


Fig. 2

DE is ⊥ to OA, therefore OEA is ⊥ both to DE and DB, and therefore to the plane BED. Hence BE is ⊥ to OA, and ∠BED = ∠A.

Triangles BDE and BEO are right-angled, the right angles being at D and E respectively.

Similarly, DF is ⊥ to OC, ∠BFD = ∠C, and △BDF and △BFO are right-angled, having right angles at D and F.

* The reader will find a wire and paper model of fig. 2 useful.

EG is drawn ⊥ to OC and therefore ∥ to DF, and DK is ⊥ to EG and therefore ∥ to OC, and KGFD is a rectangle.

Since EG is ⊥ to OC, and ED is ⊥ to OA, ∠DEG or DEK = ∠AOC = b.

From △BFO, $OF = \cos a$, and $FB = \sin a$.

From △BEO, $OE = \cos c$, and $EB = \sin c$.

From △OGE, $\frac{OG}{OE} = \cos b$, $\therefore OG = \cos b \cos c$.

$\frac{GE}{OE} = \sin b$, $\therefore GE = \sin b \cos c$.

From △BDE, $\frac{ED}{EB} = \cos A$, $\therefore ED = \sin c \cos A$.

From △BDF, $\frac{FD}{FB} = \cos C$, $\therefore FD = \sin a \cos C$.

From △EKD $\frac{KD}{ED} = \sin b$, $\therefore KD = ED \sin b$.

$\frac{EK}{ED} = \cos b$, $\therefore EK = ED \cos b$.

Now since

$$\begin{aligned} OF &= OG + GF \\ &= OG + KD \\ &= OG + ED \sin b, \end{aligned}$$

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos A. \quad \ldots \quad (Ia)$$

Similarly, $$\cos b = \cos c \cos a + \sin c \sin a \cos B, \quad \ldots \quad (Ib)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \quad \ldots \quad (Ic)$$

By means of (Ia), if two sides (b and c) and the included angle (A) are known, the other side (a) can be calculated. The other angles might then be found from (Ib) and (Ic).

3. *The distance between two places on the earth's surface.*

This can be calculated from relations (I).

Referring to fig. 3, if A and C represent the two places and B one of the earth's poles, then c is the polar distance of A (which in degrees is 90° − latitude of A), a is the polar distance of C, and B is the difference in the longitude of A and C. The great circle distance between A and C is b, and by relation (Ib)

$$\cos b = \cos c \cos a + \sin c \sin a \cos B.$$

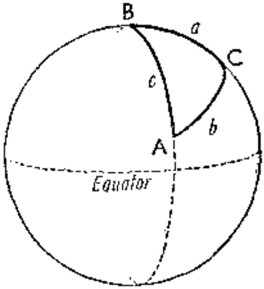


Fig. 3

Example.—Calculate the distance between New York (41° N. 74° W.) and Liverpool (53½° N. 3° W.).

Here $c = 90 - 41 = 49°$, $a = 90 - 53\frac{1}{2} = 36\frac{1}{2}°$,

$B = 74 - 3 = 71°$.

$$\begin{aligned}\cos b &= \cos c \cos a + \sin c \sin a \cos B \\ &= \cos 49° . \cos 36\tfrac{1}{2}° + \sin 49° . \sin 36\tfrac{1}{2}° . \cos 71° \\ &= \cdot 6561 \times \cdot 8039 + \cdot 7547 \times \cdot 5948 \times \cdot 3256 \\ &= \cdot 5274 + \cdot 1462 \\ &= \cdot 6736.\end{aligned}$$

From tables, $b = 47° \, 40'$.

Now one minute of arc of a great circle of the earth is one sea mile, i.e. 1° = 60 sea miles (or 69 land miles approx.). The distance between New York and Liverpool is therefore 47° × 60 + 40 = 2860 sea miles, or about 3290 land miles. Angles A and C can be calculated from (Ia) and (Ic), but the relation (II) found in the next section gives a shorter method.

4. Relations between the sines of the Angles and sines of the Sides.

Refer again to fig. 2.

From $\triangle$BDE, $\dfrac{BD}{EB} = \sin A$, $\quad \therefore$ BD $= \sin c \sin A$,

and from $\triangle$BDF, $\dfrac{BD}{FB} = \sin C$, $\quad \therefore$ BD $= \sin a \sin C$, and ∴ $$\sin a \sin C = \sin c \sin A$$

or $$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}.$$

Similarly, $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B},$$

giving $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \quad \text{(II)}$$

Returning to the example (p. 281), the angles A and C can now be calculated by (II), since

$$\sin A = \frac{\sin B}{\sin b}\sin a = \frac{\sin 71° \times \sin 36\tfrac{1}{2}°}{\sin 47° 40'}.$$

Work this out by logs and verify that A = 49° 33′. This means that for great circle sailing, a ship would leave New York in the direction 49° 33′ E. of N. Similarly, verify that C is 74° 54′.

Again, since FD = GK = GE − EK = GE − ED cos b,

$$\therefore \sin a \cos C = \sin b \cos c - \cos b \sin c \cos A. \quad \text{(III)}$$

Multiplying equation (III) by sin A and using equation (II),

$$\frac{\sin a \sin A \cos C}{\sin a \sin C} = \frac{\sin b \cos c \sin A}{\sin c \sin A} - \frac{\cos b \sin c \cos A \sin A}{\sin c \sin A}.$$

$$\therefore \sin A \cot C = \sin b \cot c - \cos b \cos A. \quad \text{(IV}a\text{)}$$

$$\sin B \cot A = \sin c \cot a - \cos c \cos B. \quad (b)$$

$$\sin C \cot B = \sin a \cot b - \cos a \cos C. \quad (c)$$

$$\sin C \cot A = \sin b \cot a - \cos b \cos C. \quad (d)$$

$$\sin A \cot B = \sin c \cot b - \cos c \cos A. \quad (e)$$

$$\sin B \cot C = \sin a \cot c - \cos a \cos B. \quad (f)$$

By means of (IV), if two sides (b and c) and the included angle (A) are known, another angle (C) can be calculated; or if two angles (A and C) and the connecting side (b) are known, another side (c) can be calculated.

If relation (IV) is arranged in the form $\cos b \cos A = \sin b \cot c - \sin A \cot C$, the following rule will help you to remember it. From the figure write down four elements in order, for example,

$$CbAc,$$

Call b and A the inner elements and C and c the outer elements; then the rule is:

Product of cosines of inners = sin inner side . cot outer side
− sin inner angle . cot outer angle,

e.g. $$\cos b \cos A = \sin b \cot c - \sin A \cot C.$$

5. The Right-angled Spherical Triangle.

When the angle C is a right angle, the relations reduce to the following.

From (Ic), $\cos c = \cos a \cos b$, (since $\cos 90° = 0$); . . (i)

from (II), $\sin a = \sin c \sin A$, (since $\sin 90° = 1$); . . (ii)

from (III), $\cos A = \dfrac{\sin b}{\sin c}\dfrac{\cos c}{\cos b}$, (since $\cos 90° = 0$);

$$\therefore \cos A = \frac{\sin B}{\sin C}\frac{\cos a \cos b}{\cos b}, \text{ from (II) and (i);}$$

$$\therefore \cos A = \sin B \cos a, \text{ (since } \sin C = 1). \quad . \ . \ . \ . \quad \text{(iii)}$$

Also, $$\cos A = \frac{\sin b \cos c}{\sin c \cos b} = \frac{\tan b}{\tan c}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{(iv)}$$

From (IVc) $\dfrac{\cot B}{\cot b} = \sin a$,

or $$\tan B = \frac{\tan b}{\sin a}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{(v)}$$

These should be verified and the corresponding relations written down when A and B and also a and b are interchanged, C still remaining a right angle; e.g. (iii) becomes $\cos B = \sin A \cos b$.

Note the formulæ

$$\sin A = \frac{\sin a}{\sin c}, \quad \cos A = \frac{\tan b}{\tan c}, \quad \tan A = \frac{\tan a}{\sin b}, \quad . \quad \text{(vi)}$$

and compare them with the corresponding formulæ for a plane triangle.

Exercise XXIV (a)

1. What does equation I become when A is (i) 90°, (ii) 180°?
2. Find side a when (i) A = 90°, and $b = c = 90°$; (ii) A = 45°, and $b = c = 90°$.

3. Calculate the angles of the spherical triangle of which $a = 36\frac{1}{2}°$, $b = 47° 40'$, $c = 49°$.

4. Find the great circle distance between two places on the line of latitude 45° N., and differing by 45° in longitude, and compare it with the arc of the circle of latitude between the two places.

5. Repeat Exercise 4 for a difference of 90° in longitude.

6. Find the great circle distance between the following places:
   (i) New York and Bristol ($51\frac{1}{2}°$ N., $2\frac{1}{2}°$ W.).
   (ii) New York and Cape Town (34° S., 18° 25′ E.).
   (iii) Cape Town and Wellington (41° S., 175° E.).
   (iv) San Francisco (38° N., 122° W.) and Wellington.
   (Remember that the cosine of angles in the second quadrant is negative.)

7. Referring to the example, p. 281, make calculations of b and A for various differences in longitude (say 15°) along the course, and plot the results against longitude.

Note.—In calculating the three angles of a spherical triangle it would be noticed that unlike the angles of a plane triangle the sum of the angles of a spherical triangle is greater than two right angles. Each angle may have a value between 0 and 180°. A triangle each of whose angles is nearly two right angles is very nearly the hemisphere. The sum of the angles of a spherical triangle lies between two and six right angles. The sum of the sides is less than the circumference of the sphere, i.e. 360°.

6. Relation between the Three Angles and One Side.

The relations for the right-angled triangle can be used to establish an important relation between all the angles and one side of the general triangle.

In the $\triangle ABC$ let great circle arc p meet AC at right angles at C′. Name the parts of $\angle B$, X and Y, and C′C, y as in fig. 4. Then from right-angled $\triangle BC'C$ by relation (i) (p. 283),

$$\cos a = \cos p \cos y$$

and by relation (iii), $\cos y = \dfrac{\cos Y}{\sin C}$,

$$\therefore \cos a = \cos p \frac{\cos Y}{\sin C}$$

or

$$\cos Y \cos p = \sin C \cos a.$$

Also by (iii), $\cos p = \dfrac{\cos C}{\sin Y}$, from which, $\sin Y \cos p = \cos C$.

From right-angled $\triangle AC'B$, by (iii),

$$\cos A = \cos p \sin X = \cos p \sin(B - Y)$$
$$= \sin B \cos Y \cos p - \cos B \sin Y \cos p.$$

$\therefore$ substituting, $\cos A = \sin B \sin C \cos a - \cos B \cos C$. . (V$a$)

Similarly, $\cos B = \sin C \sin A \cos b - \cos C \cos A$, . . ($b$)
$\cos C = \sin A \sin B \cos c - \cos A \cos B$. . . ($c$)

By relations (V), the sides can be calculated if all the angles are known; also, if two angles and the connecting side are known, the third angle can be calculated.

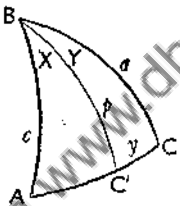


Fig. 4

Compare relations (V) and (I) and verify that if a is changed into $\pi - A$, A into $\pi - a$, and similarly for the other sides and angles, equation (Ia) gives equation (Va); thus:

$\cos(\pi - A)$
$= \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a)$;

i.e. $-\cos A$
$= (-\cos B)(-\cos C) + \sin B \sin C(-\cos a)$.

$\therefore \cos A$
$= \sin B \sin C \cos a - \cos B \cos C$.

This will help you to transform (I) into (V).

Note.—A triangle can be drawn whose sides and angles are, respectively, the supplements of the angles and sides of another triangle. Either triangle is called the *Supplemental* or *Polar* triangle of the other.

Exercise XXIV (b)

1. Use relation (V) to find A when $B = 36\frac{1}{2}°$, $C = 51\frac{3}{4}°$, and $a = 57°$, and a when $A = 104°$, $B = 36\frac{1}{2}°$, and $C = 51\frac{3}{4}°$.
2. Find A when $C = 55° 6'$, and $a = 124°$ and B has the following values: (i) 92° 25′; (ii) 90°; (iii) 75°; (iv) 60°.
3. Calculate b for the values of C, a, and B in Ex. 2.
4. Show by relations (i) and (iii) (p. 283), C being a right angle, that $\cos c = \cot A \cot B$.
5. The following are the elements of two spherical triangles; check the values by using each of the relations (I) to (V).

   (i) $A = 80°$, $B = 110°$, $C = 130°$
   $a = 56° 52'$, $b = 126° 58'$, $c = 139° 21'$,
   and
   (ii) $a = 57°$, $b = 31°$, $c = 42\frac{1}{2}°$
   $A = 104°$, $B = 36\frac{1}{2}°$, $C = 51\frac{1}{2}°$.
6. Find the great circle distance between the following places:
   Glasgow (56° N., $4\frac{1}{4}$° W.) and New York,
   London ($51\frac{1}{2}$° N. 0°) and Calcutta (22° 34′ N., 88° 24′ E.),
   London and Moscow (55° 45′ N., 37° 36′ E.),
   Calcutta and Port Darwin (12° 20′ S., 130° 50′ E.),
   London and Port Darwin,
   Honolulu (21° 20′ N., 157° 51′ W.) and Yokohama (35° N., 139° 30′ E.),

   and the great circle direction from the first named and also the direction of arrival.
7. Take the great circle course from New York to Cape Town and calculate by (V) the direction at various longitudes along the course, say at intervals of 15°. Find also the distance from New York and Cape Town for these longitudes. Plot the results against longitude and so construct a graph like that shown in fig. 5. See also fig. 6.
8. In a manner similar to that of Ex. 7, make out great circle air courses from

   London to Calcutta (22° 34′ N., 88° 24′ E.).
   Calcutta to Sydney (33° 52′ S., 151° 12′ E.).
   London to Cairo (30° 2′ N., 31° 15′ E.).

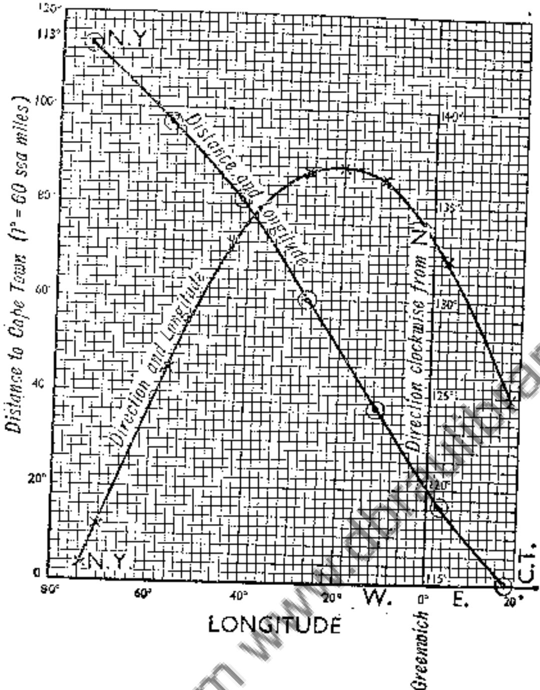

N.Y
Distance and Longitude
Direction and Longitude
Direction clockwise from N
Distance to Cape Town (1° = 60 sea miles)
N.Y
C.T.
W.
E.
LONGITUDE
Greenwich

Fig. 5

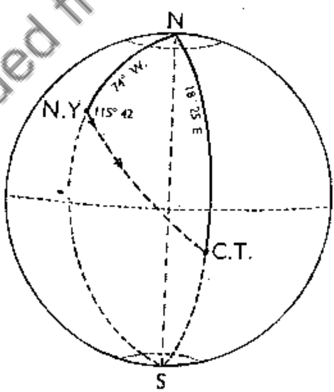

N
N.Y
74° W.
115° 42
18° 25 E
C.T.
S

Fig. 6

CHAPTER XXV

FUNCTIONAL NOTATION, VARIATION, EXPANSION OF BINOMIALS, APPROXIMATIONS

1. Functional Notation.

A special notation is used to denote functions. The signs most generally used are, $f(x)$, $F(x)$ and $\phi(x)$, the letter in the brackets indicating the variable.

Thus, $f(x) = x^2 - 3x + 2$ means that the expression $x^2 - 3x + 2$ is a function of x. Similarly, $f(x) = ax^2 + bx + c$ means that the expression has to be regarded as a function of x only. That is, x is the only symbol which changes in value; the others being therefore constants.

The value of a function when a definite value, say 3, is given to the variable is sometimes referred to as $f(3)$.

Thus, if $f(x) = x^2 - 3x + 2$, then $f(3) = 9 - 9 + 2 = 2$.

Change of Variable.

Example.—Represent $f(x) = x^2 - 3x + 2$ as a function of z, given that $z = x + 1$.

Since $z = x + 1$, $x = z - 1$.

Substitute this value of x in the given expression; then

$$f(z) = (z - 1)^2 - 3(z - 1) + 2,$$

$$\text{i.e. } f(z) = z^2 - 5z + 6.$$

Compare this with the change in the position of the axis of y (p. 207).

Exercise XXV (a)

1. If $f(x) = x^2 - 3x + 5$, find $f(2)$, $f(0)$ and $f(-3)$.
2. If $f(x) = 2x^2 - x + 1$, represent the expression as a function of z when $z = (x - 1)$.
3. If $f(x) = (2x + 1)(2x - 1) - (x + 1)$, find $f(y)$ when $y = (x - 1)$, and solve the equation $f(y) = 3$.
4. If $f(x) = \dfrac{(x + 3)(6x - 7)}{(3x + 5)(2x - 5)}$, find $f(-2)$, and state for what values of the variable x, $f(x) = 0$.

5. If d is a function of v such that

$$d = \frac{v^2 - u^2}{2a},$$

and v is a function of t such that

$$v = u + at,$$

express d as a function of t.

6. Express $\frac{3x + 11}{x^2 + 7x + 12}$ as the sum of two simple fractional functions of x (see Exercise XV (F), No. 31).

7. Express $\frac{6x - 17}{6x^2 + 5x - 6}$ as the difference between two simple fractional functions of x.

8. If $f(x) = \sin 2x \cos x$, find $\frac{f(30°)}{f(45°)}$.

9. If $f(x) = 3 \tan (x + 15°)$, find $f(0°)$ and $f(30°)$.

2. Variation.

Generally, one quantity is said to vary with another when one is so dependent upon the other that it changes when a change is made in the other. The law defining the change may be simple or may be complex.

EXAMPLES.

(1) The perimeter of a square depends on the length of the side of the square.

(2) The area of a circle depends on the length of the radius.

(3) The weight of a liquid depends upon the volume of the liquid, the temperature being constant.

(4) The weight of a bag of sovereigns depends upon the number of sovereigns the bag contains.

(5) The amount of interest due depends upon the principal invested, the time, and the rate paid.

As to how these quantities vary is a matter for consideration. At present we need remark only that one quantity is a function of the other.

(1) *Direct Variation.* One quantity is said to vary directly as another when one is so dependent upon the other that the ratio of any two values of one quantity is equal to the ratio of the

corresponding values of the other. Or if when one is changed, the other is changed in the same ratio.

E.g. the weight of water varies as the volume taken.

Thus: 10 c. ft. weigh 625 lb.,
6 c. ft. weigh 375 lb.,
2 c. ft. weigh 125 lb.

The ratio $\frac{10}{6}$ will be found equal to the ratio $\frac{625}{375}$,
also $\frac{6}{2}$ will be found equal to the ratio $\frac{375}{125}$,
and similarly for other corresponding values.

If V represents the volume of water and W the weight of this volume, then W varies as V. The sign $\propto$ stands for "varies as", and thus we write:

$$W \propto V.$$

Now, returning to our example, if we divide each weight by the corresponding volume, we obtain the same result in each case, thus:

$$\frac{625}{10} = 62{\cdot}5, \quad \frac{375}{6} = 62{\cdot}5, \quad \frac{125}{2} = 62{\cdot}5.*$$

Looking at the reverse operation, we see that to get the numerical value of the weight from that of the volume, we must multiply the volume by 62·5,

$$\text{i.e. } W = 62{\cdot}5V.$$

The number 62·5 is called the constant, and is obtained by dividing one value of the quantity by the corresponding value of the other.

Hence, when we meet a statement like

$$W \propto V,$$

we can at once write it in the form of an equation, thus:

$$W = KV,$$

where K is a constant which can be obtained by dividing a value of W by the corresponding value of V.

For a more general proof, suppose V_1, V_2, V_3, etc., represent values of one quantity, and W_1, W_2, W_3, etc., represent corresponding values of the other; then, since W varies as V,

$$(1)\ \frac{V_1}{V_2} = \frac{W_1}{W_2}, \qquad (2)\ \frac{V_2}{V_3} = \frac{W_2}{W_3}, \qquad (3)\ \frac{V_1}{V_3} = \frac{W_1}{W_3}, \text{ etc.}$$

* These fractions are not ratios in the strict sense of the word, because the terms are not of the same kind. They merely represent quotients.

Transposing, we have that

$$\frac{W_2}{V_2} = \frac{W_1}{V_1}, \qquad \frac{W_3}{V_3} = \frac{W_2}{V_2}, \qquad \frac{W_3}{V_3} = \frac{W_1}{V_1}.*$$

Examining these, it is evident that all these fractions are equal to a constant number, which we have called K,

$$\text{i.e.} \quad \frac{W_1}{V_1} = \frac{W_2}{V_2} = \frac{W_3}{V_3}, \text{ etc.} = K.$$

Graph the numbers given as corresponding values of weight and volume, and draw your conclusions from the appearance of the graph.

You will find that the graph is a straight line of gradient K. Hence W is a linear function of V.

(2) One quantity may vary as the inverse of another, or the square, cube, square root, etc., of another.

For example, in your Mensuration or Science lessons, you will find that the area of a circle varies as the square of the radius; that the volume of a sphere varies as the cube of the radius; that the time of the swing of a simple pendulum varies as the square root of the length; that the volume of a gas at constant temperature varies inversely as the pressure. One quantity is said to vary *conjointly* as a number of others when it varies as their product. Thus, the value of a bar of gold varies conjointly as its length, breadth, and thickness.

Example.—A pendulum 100 cm. long takes 2 sec. to swing to and fro. Find the time for a pendulum 36 cm. to swing to and fro.

Let t represent the time of swing, say in sec., and l the length, say in cm., of any pendulum; then t varies as $\sqrt{l}$;

$$\therefore\ t = K\sqrt{l} \quad \text{and} \quad \therefore\ K = \frac{t}{\sqrt{l}}.$$

To find K, make use of the given value of t (2 sec.) for the pendulum of length 100 cm.

$$K = \frac{2}{\sqrt{100}} = \frac{2}{10} = \cdot 2.$$

$\therefore$ the equation connecting t and l is

$$t = \cdot 2\sqrt{l}.$$

* These fractions are not ratios in the strict sense of the word, because the terms are not of the same kind. They merely represent quotients.

To find the time for a pendulum 36 cm. long,

$$t = \cdot 2\sqrt{36} = 1{\cdot}2 \text{ sec.}$$

Exercise XXV (b)

1. If $a \propto b$, and $b = 12$ when $a = 2$, find the equation connecting a and b, and find a when b is 4·5.
2. If $x \propto y$ and $y \propto z$, show that $x \propto z$.
3. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.
4. The weight of a cable of given thickness and material varies as the length. If a length, 120 yd., of this cable weighs 385 lb., find the weight of a length 3 miles. Find also the weight per mile.
5. Show that the volumes of similar cones vary as the cube of their altitudes.
6. Show that the volumes of spheres vary as the cube of their radii.
7. The sag in a telegraph wire varies directly and conjointly as the length and the weight, and inversely as the horizontal tension.

   When the weight is 2 oz. per foot, the length 80 ft. and the horizontal tension 150 lb., the sag is 8 in. Find the sag when the weight is $1\frac{1}{2}$ oz. per foot, the length 20 yd. and the tension 100 lb.
8. If $y^n = \frac{k}{x}$, find n and k, given that x is 5 when y is 10, and is 11 when y is 8.
9. The period of a planet, that is, the time it takes to make one revolution round the sun, is found to vary as the square root of the cube of its distance from the sun. Knowing the period and distance of the earth, find the distance of Jupiter, the period of which is observed to be 11·86 years.
10. In printing gas-light photographic papers, the time of exposure varies as the square of the distance of the plate from the source of light. If for a distance 8 in. the time is 6 sec., what exposure is necessary for a distance 18 in.? For what distance would the time be 12 sec.?

3. Rapid Expansions.*

We have seen that

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$
$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3,$$
$$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4.$$

Notice that the terms run in descending powers of a and ascending powers of b, and that when the sign between the two given terms is $+$ the signs of the expansion are all $+$, and that when $-$, the signs of the expansion run alternately $+$ and $-$, the first sign being $+$, although not shown.

Look at the last example, and verify this rule for finding the coefficient of a term, say the third term, from the preceding term. Examine the second term. Multiply its coefficient (4) by the index of the descending power, i.e. the index 3 of a^3, and divide the product by the number of the term, i.e. being the second term by 2. The result is the coefficient of the next term. Thus, $\frac{4 \times 3}{2} = 6$.

Try the second term the same way. Remember the coefficient of the first term is 1.

To expand $(a - b)^5$.

First write down the terms without their coefficients in descending powers of a and ascending powers of b.

Thus, $(a - b)^5$,	a^5	$-$	a^4b	$+$	a^3b^2	$-$	a^2b^3	$+$	ab^4	$-$	b^5.
	1st		2nd		3rd		4th		5th		6th

Then calculate the coefficients

$$\frac{1 \times 5}{1} \quad \frac{5 \times 4}{2} \quad \frac{10 \times 3}{3} \quad \frac{10 \times 2}{4} \quad \frac{5 \times 1}{5}$$

$$(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

The expansion can be readily checked by putting $a = 1$ and $b = 1$; then each side should equal 0.

The coefficients of the expanded powers of $(a + b)$ can be

* The general formula for the expansion of a binomial is given in Chapters XXVI and XXVII.

arranged so as to show how one set of coefficients can be obtained from those of the next lower power. Thus:

	Coefficients in order
$(a + b)^0$	1
$(a + b)^1$	1 1
$(a + b)^2$	1 2 1
$(a + b)^3$	1 3 3 1
$(a + b)^4$	1 4 6 4 1
$(a + b)^5$	1 5 10 10 5 1
etc.	

The first and last coefficients are always unity. The brackets indicate that the coefficients of one expanded power when added in successive pairs give the intermediate coefficients of the next power.

The arrangement suggests a triangle, and is known as Pascal's triangle—Pascal being the name of the discoverer.

The reason underlying Pascal's triangle is readily understood if two successive powers are compared.

Take, for example, $(a + b)^4$ and $(a + b)^5$.

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$

$$\begin{aligned}(a + b)^5 &= (a + b)(a + b)^4 \\ &= (a + b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\ &= b(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\ &\qquad + a(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\ &= a^5 + (1 + 4)a^4b + (4 + 6)a^3b^2 + (6 + 4)a^2b^3 \\ &\qquad + (4 + 1)ab^4 + b^5.\end{aligned}$$

This arrangement shows how the coefficients of the expansion of $(a + b)^5$ are obtained from those of the expansion of $(a + b)^4$.

Exercise XXV (c)

1. Expand $(x + y)^4$ and $(x + y)^6$.
2. Expand $(x - y)^4$ and $(x - y)^6$.
3. Expand $(a - b)^3$, $(a + b)^7$, $(a - b)^5$.
4. Expand $(a^2 - b^2)^5$.

5. Expand $(2a - 3b)^4$. (Take $(x - y)^4$ as a pattern, and substitute in the expansion, $2a$ for x and $3b$ for y.)
6. Find $(2a - 3b)^7$.
7. Expand $\{(a + b) + (b + c)\}^3$.
8. Find $\{(a + b) - (c + 2)\}^4$.
9. Find the fifth power of $(x^2 + y^2 - z^2)$.
10. In the expansion of $(a + b)^5$, put a and b each equal to 1 and then find the sum of the coefficients.
11. Find, correct to the second place of decimals, $(1{\cdot}005)^8$.
12. Expand $(1 - 2a)^{10}$.

4. Approximations.

A small quantity, say a small increase or decrease, is usually denoted by the symbol δ or δx. The latter does not mean δ times x, but is equivalent to a single symbol.

Consider the expansion $(1 + x)^2 = 1 + 2x + x^2$.

If x is small compared with unity, i.e. if x is a small fraction, x^2, being a fraction of a fraction, will be smaller still. Thus, if x is 0·1, then x^2 is 0·01. Generally, the square, and therefore the higher powers of very small numbers, may be neglected.

If we write δ for x, we have

$$(1 + \delta)^2 = 1 + 2\delta \text{ approximately.}$$

Similarly, if n represents any power,

$$(1 + \delta)^n = 1 + n\delta.$$

EXAMPLES.

$$(1{\cdot}0012)^2 = (1 + {\cdot}0012)^2 = 1 + {\cdot}0012 \times 2$$
$$= 1{\cdot}0024 \text{ approx.,}$$
$$(1{\cdot}0006)^7 = 1 + {\cdot}0006 \times 7 = 1{\cdot}0042 \text{ approx.}$$

The same is true for roots. Thus:

$$\sqrt{1 + \delta} = (1 + \delta)^{\frac{1}{2}} = 1 + \tfrac{1}{2}\delta.$$

EXAMPLE. $\sqrt[3]{1{\cdot}0026} = 1 + \dfrac{{\cdot}0026}{3} = 1{\cdot}00087$ approx.,

$$\sqrt[5]{0{\cdot}9995} = (1 - {\cdot}0005)^{\frac{1}{5}} = 1 - \frac{{\cdot}0005}{5}$$
$$= 1 - {\cdot}0001$$
$$= {\cdot}9999 \text{ approx.}$$

Applications.

1. *Expansion.*

(i) *Area.* A metal plate has the shape of a square, and its edge is of unit length. When its temperature is raised one degree, each edge increases in length by a small amount δ, called the coefficient of linear expansion. Find the increase in area.

Area before the temperature is raised = 1 unit of area.

„ after „ „ „ $1^\circ = (1 + \delta)^2$ units of area.

The increase in area for 1° rise in temperature $= (1 + \delta)^2 - 1$

$= 2\delta + \delta^2$.

Now δ is very small, e.g. for iron, per °C., $\delta = \cdot 0000117$; for copper, ·000017; and therefore δ^2 can be neglected. *The small corner square of the figure shows that δ^2 is small compared with 2δ.*

It follows that the coefficient of surface expansion is approximately 2δ, i.e. approximately twice the coefficient of linear expansion.

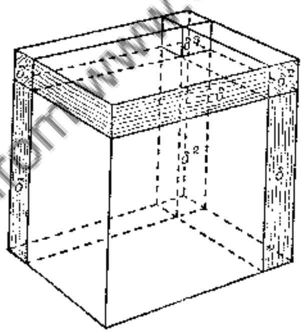


Fig. 1

(ii) *Volume.* A metal cube of unit edge, when raised one degree in temperature, has each edge increased in length by a small amount δ. Find the increase in volume.

Volume before temperature raised = 1 unit of volume.

„ after „ „ $1^\circ = (1 + \delta)^3$ units of volume.

Increase in unit volume for 1° rise $= (1 + \delta)^3 - 1$ „ „

$= 3\delta + 3\delta^2 + \delta^3$ „ „

δ being small, the terms $3\delta^2$ and δ^3 are negligible (fig 1).

It follows that the coefficient of volume expansion is 3δ, i.e. three times the coefficient of linear expansion.

For t° rise in temperature, the surface expansion is $2t\delta$ per unit area, and the volume expansion $3t\delta$ per unit volume.

2. The volume (V) of the wall or shell of a hollow sphere of external radius R, and thickness t, is:

$$\begin{aligned} V &= \tfrac{4}{3}\pi R^3 - \tfrac{4}{3}\pi(R - t)^3 \\ &= \tfrac{4}{3}\pi(3R^2t - 3Rt^2 + t^3) \\ &= 4\pi R^2 t\left(1 - \frac{t}{R} + \frac{t^2}{3R^2}\right). \end{aligned}$$

If t is small compared with R this reduces to:

$$\begin{aligned} V &= 4\pi R^2 t \\ &= \text{surface} \times \text{thickness}. \end{aligned}$$

5. Small Angles.

In fig. 2, $\angle AOP$ is a small angle, the number of radians in which is δ.

The value of δ is $\dfrac{\text{arc PA}}{OP}$; also $\sin\delta = \dfrac{PN}{OP}$, $\cos\delta = \dfrac{ON}{OP}$, and $\tan\delta = \dfrac{PN}{ON}$.

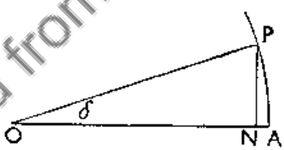


Fig. 2

When δ is small the difference between arc PA and perpendicular PN is small, as is also the difference between OA and ON.

Imagine the angle to become smaller by moving P towards A. Then N will also move towards A. As a result PN will become more and more nearly equal to arc PA, and ON to OA and OP.

Equality will only be reached when P actually reaches A, when, of course, the angle is 0.

It is seen then that when δ is very small,

$$\text{arc PA} \approx PN$$

and

$$ON \approx OP.$$

It follows that, if the angle δ is measured in radians,

$$\sin\delta \approx \delta,\ \tan\delta \approx \delta,\ \text{and}\ \cos\delta \approx 1,$$

where the symbol $\approx$ means "is approximately equal to".

This is confirmed by reference to tables.

$3^\circ = \cdot0524$ radians, $\tan 3^\circ = \cdot0524$, $\sin 3^\circ = \cdot0523$, $\cos 3^\circ = \cdot9986$.
$1^\circ = \cdot0175$ radians, $\tan 1^\circ = \cdot0175$, $\sin 1^\circ = \cdot0175$, $\cos 1^\circ = \cdot9998$.

Exercise XXV (d)

1. The dimensions of a rectangular plate of copper at 0° C. are 18 in. by 15 in. Find the area of its surface at 50° C.
2. A cylinder of copper has a diameter of 12 cm. at 15° C. Find the area of one end at 100° C. If its length at 15° C. is 15 cm., find its volume at 100° C.
3. The volume of a flask is 250 c.c. at 0° C. What will be its volume at 100° C.? (Coefficient of linear expansion of glass ·000009.)
4. The diameter of a spherical glass bulb is increased by 1 per cent. By what percentage is its capacity increased?
5. The capacity of a glass flask at 0° C. is 1 litre. If it contains air, what volume will escape when the flask is heated to 100° C.? (Coefficient of expansion of air is $\frac{1}{273}$ per degree C.)
6. Find approximately:

$$\sqrt[5]{1\cdot0003},\quad \sqrt[3]{0\cdot997},\quad \sqrt[3]{1002\cdot1},$$
$$(1\cdot002)^2,\quad (\cdot996)^4,\quad (10\cdot06)^2.$$

7. By how much must the temperature of a sheet of iron be raised in order that its surface area may be increased by 1 per cent?
8. At what temperature will a rod of copper 199 cm. long at 15° C. and a rod of iron 200 cm. long at 15° C. have the same length?
9. If δ is a small fraction, show that $\dfrac{1}{1+\delta} = 1 - \delta$ approximately.
10. What is the volume of the material of a hollow sphere of diameter 10 in., the thickness of the material being $\frac{1}{16}$ in.?
11. What are the errors per cent in writing δ for $\sin\delta$ and $\tan\delta$, and 1 for $\cos\delta$ when δ is (i) 5°, (ii) $\frac{1}{2}$°?
12. Find (i) $\tan\delta\ \tan x$; (ii) $\sin\delta\ \sin x$; (iii) $\cos\delta\ \cos x$, when $\delta = 0{\cdot}1^\circ$ and $x = 60^\circ$. What is the value of each when $\delta = 0$?

Exercise XXV (e)

1. The area of a sector of a circle is $\frac{1}{2}Ra$, where a = the length of the arc and R the radius. What does this formula become when the sector is a semicircle? What, therefore, is the area of the whole circle?

2. The area of the curved surface of a spherical cap is $2\pi Rt$, where R is the radius of the sphere and t the altitude of the cap. What does this become when the cap is a hemisphere? What then is the surface of the whole sphere?

3. The volume of an ellipsoid is $\frac{4}{3}\pi abc$, where a, b and c are its semi-axes. What does this become for the sphere?

4. The volume of the cap of a sphere is $\frac{1}{6}\pi t(t^2 + 3r^2)$ when t is the altitude and r the radius of the base of the cap. Apply this formula to a hemisphere, and to a whole sphere.

5. The area of a segment of a circle is given approximately by the formula $\frac{h}{6b}(4b^2 + 3h^2)$, where h is the altitude and b the base of the segment. Apply this to the semicircle, and find the error per cent.

6. Another formula for the area of a segment of a circle is $\frac{2}{3}h(b + \frac{4}{3}c)$, in which c is the chord of the semi-arc. Apply this to the semicircle, and find the error per cent.

7. The curved surface of the frustum of a cone is $\pi s(R + r)$, where s is the slant height, and R and r the radii of the base and top. What does this become for the full cone?

8. The volume of the frustum of a cone is $\frac{1}{3}\pi h(R^2 + Rr + r^2)$, where h is the altitude of the frustum. What does this become for the full cone? Apply the given formula to a cylinder.

CHAPTER XXVI

PROGRESSIONS, SERIES, PERMUTATIONS AND COMBINATIONS, BINOMIAL THEOREM

1. Progressions.

By a progression we mean a series of numbers which proceeds in order according to some law.

The two simplest progressions are *Arithmetical Progression* and *Geometrical Progression.*

(i) *Arithmetical Progression* (A.P.)

In an Arithmetical Progression, the terms proceed by equal added amounts (or differences).

The added amount may be positive or negative.

EXAMPLES.

(i) 2, 5, 8, 11, 14, 17, etc.

The above numbers form an Arithmetical Progression.

Beginning from the first number, 2, the second, 5, is obtained by adding 3 to the first. Similarly, the third number, 8, is obtained by adding 3 to the second, and so on.

The added amount, or *common difference* as it is more usually called, is found by taking any term and subtracting from it the preceding term.

(ii) 10, 5, 0, -5, -10, etc.

The common difference in this A.P. is -5. Verify this statement.

(iii) $3a$, $7a$, $11a$, $15a$, etc.

This is an A.P. having a common difference $4a$. Verify this.

EXERCISE XXVI (A)

1. Write down three more terms to each of the given examples.
2. Write down a few terms of the A.P. of which the first term is $3a$ and the common difference $-4a$. Contrast this progression with Example iii.

3. Write eight terms of the following Arithmetical Progressions:

(i)	First term	6,	common difference 4.
(ii)	,, ,,	6	,, ,, -4.
(iii)	,, ,,	-6	,, ,, 4.
(iv)	,, ,,	-6	,, ,, -4.
(v)	,, ,,	1	,, ,, -4.
(vi)	,, ,,	0	,, ,, -2.

4. Construct the A.P. of which the first term is a and the common difference d. Compare the coefficient of d in any term with the number of that term.

(ii) *Geometrical Progression* (G.P.)

In a Geometrical Progression, the terms proceed by a constant ratio. In other words, the ratio any term bears to the preceding term is the same throughout the series.

The constant ratio may be positive or negative.

EXAMPLES.

(i) 2, 6, 18, 54, 162, etc.

The above numbers form a Geometrical Progression.

Beginning from the first number, 2, the second, 6, is obtained by multiplying the first by 3. Similarly, the third, 18, is obtained by multiplying the second by 3, and so on.

The common ratio is found by dividing any term by the preceding term.

Contrast this progression with the A.P., Example (i), p. 300.

(ii) 12, -3, $\frac{3}{4}$, $-\frac{3}{16}$, etc.

The common ratio of this G.P. is $\frac{-3}{12} = -\frac{1}{4}$. Verify this statement.

(iii) $3a$, $6a^2$, $12a^3$, $24a^4$, etc.

This is a G.P. having a common ratio $2a$. Verify this.

EXERCISE XXVI (B)

1. Extend each of the series in Examples (i) (ii) (iii), by three terms.
2. Write down a few terms of the G.P. of which the first term is $3a$ and the common ratio $-2a$. Contrast this progression with Example iii.

3. Write down six terms of the following Geometrical Progressions:

(i) First term 1, common ratio -2.
(ii) „ „ 1, „ „ $\frac{1}{2}$.
(iii) „ „ -2, „ „ $-\frac{1}{3}$.

4. Construct the Geometrical Progression of which the first term is a and the common ratio r. Compare the index of the power of r of any term with the number (in order) of that term.

5. Determine whether the following series are in Geometrical or Arithmetical Progression. Give reasons in each case.

(i) 3, 6, 9, 12, etc. (ii) 3, -6, 12, -24, etc.
(iii) -4, -2, 0, etc. (iv) -4, -2, -1, etc.

2. Graphical Representation.

(i) *Arithmetical Progression*

If the terms of an A.P. are plotted against the order of the terms, then, since they proceed by equal added amounts, the plotted points lie in a straight line (see p. 141).

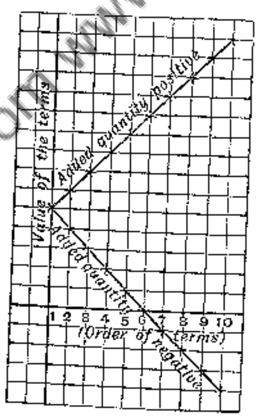


Fig. 1

This straight line * has an up gradient if the added quantity is positive and a down gradient if negative (fig. 1).

* *If the line is drawn it is not a graph in the sense in which we have already used the name, since the portions between the plotted points have no significance. The line serves to show only the relative position of the plotted points.*

(ii) *Geometrical Progression*

If we plot the terms of a G.P. against the order of the terms, the points lie on a curve.

CASE 1.—When the common ratio is positive and greater than unity, the terms increase and the curve diverges from the axis of x (fig. 2).

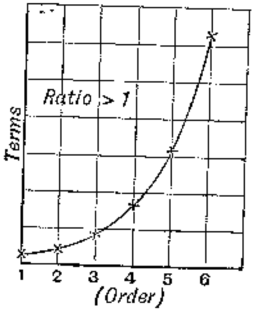


Fig. 2

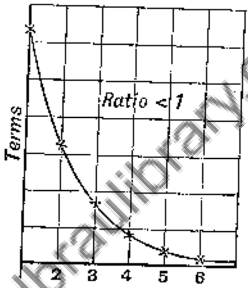


Fig. 3

CASE 2.—When the common ratio is positive and less than unity (fractional), the successive terms decrease and the curve gradually approaches the axis of x (fig. 3).

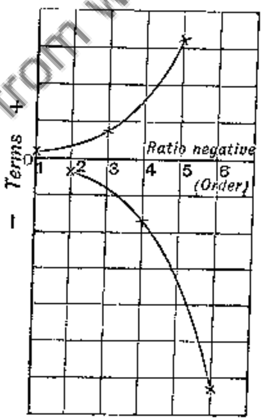


Fig. 4

CASE 3.—When the common ratio is negative the terms are alternately positive and negative. Such a series can be regarded as consisting of two series, one positive and the other negative (fig. 4).

The common ratio of each set is the square of the ratio of the series. Prove this.

EXERCISE.—Apply the graphical method to Ex. XXVI(E), No. 5.

3. Arithmetical Progression.

(i) *General Term*

In Exercise XXVI (A), No. 4, you should have found that in the A.P., the first term of which is a and the common difference d, the coefficient of d is one less than the number of the term.

If we call any term the nth term, n representing the number of the term in order of succession, then

The nth term is $\{a + (n - 1)d\}$.

This is called the general term, or the general expression for any term. From it any term can be found without writing down the terms which precede it.

EXAMPLE.—Find the 35th term of the series:

$$1, \quad 5, \quad 9, \quad 13, \quad \text{etc.}$$

Since the series proceeds by equal added amounts, it is an A.P.
The first term is 1 and the common difference 4.
The 35th term is $1 + (35 - 1)4 = 1 + 34 \times 4 = 137$.

(ii) *Means*

The terms between any two chosen terms of a series are called **Means.** In an Arithmetical Progression, such terms are called Arithmetic Means, and when there is only one term between the chosen terms it is called the Arithmetic Mean of the other two.

To insert a given number of Arithmetic Means (A.M.S) *between given numbers.*

EXAMPLE.—Insert 6 A.M.S between 2 and -26.

Let d = the common difference.
Then the terms are:

$$2, \quad (2 + d), \quad (2 + 2d), \quad (2 + 3d), \quad (2 + 4d), \quad (2 + 5d),$$
$$(2 + 6d) \quad \text{and} \quad (2 + 7d) \text{ or } -26.$$

It is seen that -26 is the 8th term.

Hence
$$2 + 7d = -26.$$
$$7d = -28.$$
$$d = -4.$$

The means are therefore:

$$-2, \quad -6, \quad -10, \quad -14, \quad -18, \quad \text{and} \quad -22.$$

The *Arithmetic Mean* (A.M.) of two numbers is half their sum. Let the numbers be a and b and their A.M., M. Then a, M and b form an A.P.

Hence
$$M - a = b - M,$$
$$2M = a + b,$$
$$M = \frac{a + b}{2}.$$

(iii) *The Sum of a Number of Terms in* A.P.

Consider first a numerical series, say 2, 5, 8, 11, 14, 17, etc., and let us find the sum of, say, 7 terms.

Represent the sum by S; then

$$S = 2 + 5 + 8 + 11 + 14 + 17 + 20$$

and reversing the terms,

$$S = 20 + 17 + 14 + 11 + 8 + 5 + 2$$

By adding,
$$2S = 22 + 22 + 22 + 22 + 22 + 22 + 22$$
$$= 22 \times 7;$$
$$S = \frac{22 \times 7}{2},$$
i.e. $S = 77.$

Observe that the sum is half the product of the sum of the first and last terms and the number of terms,

$$\text{i.e.} \quad S = \frac{(\text{first} + \text{last}) \times \text{number of terms}}{2}.$$

The General Formula.

Let the series be:

$$a, \quad (a + d), \quad (a + 2d), \ldots (a + \overline{n-1}d);$$

then, if S_n represents the sum of n terms,

$$S_n = a + (a+d) + (a+2d) + \ldots + (a+\overline{n-1}d)$$

and

$$S_n = (a+\overline{n-1}d) + (a+\overline{n-2}d) + (a+\overline{n-3}d) + \ldots + a$$

Adding,

$$2S_n = (2a+\overline{n-1}d) + (2a+\overline{n-1}d) + (2a+\overline{n-1}d) + \ldots + (2a+\overline{n-1}d)$$

← n terms all alike →

$$= n(2a + \overline{n-1}d);$$
$$S_n = \frac{n}{2}(2a + \overline{n-1}d).$$

As before, the result may be stated in the form:

$$S_n = \frac{n}{2}(\text{first term} + \text{last term}).$$

Exercise XXVI (c)

1. Find the 21st term of the series: 1, −3, −7, −11, etc.
2. The first term of an A.P. is 2, and the tenth 29. Write down the first six terms.
3. Show that the series formed by adding each term of an A.P. to the succeeding term is an A.P.
4. Insert three arithmetic means between 3 and 18.
5. Insert four arithmetic means between 5 and −10.
6. Find the sum of the first 25 whole numbers.
7. Establish a formula for the sum of the first n integers.
8. Find the sum of the first 20 odd numbers.
9. Find the sum of the first 20 even numbers.
10. Establish formulæ for the first n odd, and for the first n even numbers.
11. Prove that the number of dominoes required for a set is the sum of the following series:

    $$1, 2, 3 \ldots x, (x + 1),$$

    where x denotes the highest number used (i.e. the highest domino is double x).

    Find the number of dominoes required for a set, the highest domino of which is double six.
12. Find the sum of 20 terms of the series whose nth term is $(6n - 2)$.
13. Plot the terms of an A.P., and on the same figure show a rectangle, the area of which represents the sum of the terms.
14. If a, b, c and d are consecutive terms of an A.P., show that $bc - ad = 2(b - c)^2$.

4. Geometrical Progression.

(i) *General Term*

Your answer to Exercise XXVI (B), No. 4, should be that the index of r is always one less than the number of the term.

Thus the nth term of the G.P. a, ar, ar^2, ar^3, etc., is ar^{n-1}.

This enables us to write down any term without determining the preceding terms.

Example.—Find the tenth term of the series:

$$3, \quad -6, \quad 12, \quad -24, \quad \text{etc.}$$

Here a is 3 and r is $-6/3$, i.e. -2.

Hence the tenth term is $3 \times (-2)^9 = 3 \times -512 = -1536$.

If logarithms are used in such exercises as the above, it must be remembered that the accuracy of the result depends upon the range of the logarithms used.

Applying logarithms to the formula for the general term, we have:

$$\log(n\text{th term}) = \log a + (n-1)\log r.$$

(ii) *Geometric Means*

The terms between any two terms of a Geometrical Progression are called Geometric Means.

When there are three terms only, the middle term is called the Geometric Mean (G.M.) of the other two.

To insert a given number of Geometric Means between given numbers.

Example.—To insert three Geometric Means between 2 and 162.

Let r be the common ratio; then the terms are:

$$2, \quad 2r, \quad 2r^2, \quad 2r^3 \quad \text{and} \quad 2r^4, \text{ or } 162.$$

Hence
$$2r^4 = 162,$$
$$r^4 = 81,$$
$$r = \pm 3.$$

The means are, therefore, 6, 18, 54,

or

-6, 18, -54.

The *Geometric Mean* of two numbers is the square root of their product.

Let G represent the G.M. of a and b.

Then a, G and b are in G.P.

Hence
$$\frac{G}{a} = \frac{b}{G},$$
$$G^2 = ab,$$
$$G = \sqrt{ab}.$$

Contrast this with the A.M. of a and b.

(iii) *Geometrical Construction*

To find the G.M. *of two given straight lines.*

Let the lengths of the lines be x and y units respectively. Referring to fig. 5, AB = x and BC = y, AC = $(x + y)$. Bisect AC at D; then DC = $\frac{1}{2}(x + y)$ and DB = $\frac{1}{2}(x - y)$. With centre D and radius DC, describe a semicircle on AC. From B, erect a perpendicular to meet the curve at P. Join D and P.

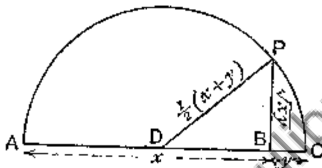


Fig. 5

Then $\quad DP = \frac{1}{2}(x + y)$.

Now $\quad PB^2 = AB \times BC = xy$ (see Ch. XI, **5**, 3).

Hence $\quad PB = \sqrt{xy}$.

Observe that DP is the A.M. of x and y. It follows, since the hypotenuse of a right-angled triangle is its greatest side, that in general the A.M. of two numbers is greater than the G.M. Under what conditions are the two means equal?

(iv) *The Sum of a Number of Terms in* G.P.

Let the terms be a, ar, ar^2, ar^3, etc., and let the sum of n terms be represented by S_n.

Then $\quad S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

Multiply by r, $\quad rS_n = \quad ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$

Subtract, $\quad S_n - rS_n = a \ldots\ldots\ldots\ldots - ar^n$

$$S_n(1 - r) = a(1 - r^n),$$

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

When r is greater than 1, the above formula is better written in the form:

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

Hence, to find the sum, extend the series by one term, and divide the difference between this term and the first term by the difference between the common ratio and unity.

Example.—Find the sum of five terms of the series:

$$6, \quad -2, \quad \tfrac{2}{3}, \quad -\tfrac{2}{9}, \quad \text{etc.}$$

The numbers are in G.P.; the common ratio is $\frac{-2}{6} = -\frac{1}{3}$.

$$S_5 = \frac{6\{1 - (-\frac{1}{3})^5\}}{1 - (-\frac{1}{3})} = \frac{6(1 + \frac{1}{243})}{1\frac{1}{3}}.$$
$$= 4\tfrac{14}{27}.$$

Verify this result by Arithmetic.

The following graphical representation of the sum is interesting:

Let $P_1, P_2, P_3, \ldots P_n$, represent the terms of a G.P. (fig. 6); then

$$1P_1 = a,\ 2P_2 = ar,\ 3P_3 = ar^2 \ldots nP_n = ar^{n-1}.$$

Join P_2, P_1, and produce the straight line P_2P_1 to cut the axis of n at Q.

Draw P_1M parallel to Qn; then $P_1M = 1$ and $MP_2 = ar - a$, or $a(r - 1)$.

Now triangle P_1Q1 is similar to triangle P_2P_1M; therefore

$$\frac{1Q}{1P_1} = \frac{P_1M}{MP_2},$$

$$1Q = 1P_1 \times \frac{P_1M}{MP_2}$$

$$= a \times \frac{1}{a(r-1)}$$

$$= \frac{1}{r-1}.$$

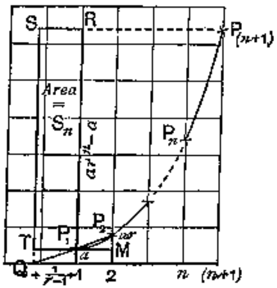


Fig. 6

The sum of n terms is $\frac{ar^n - a}{r - 1}$ or $(ar^n - a) \times \frac{1}{r-1}$.

Draw QS parallel to 1R.

Take the next term, namely, the $(n + 1)$th term. Its value is ar^n. Let P_{n+1} denote its position on the graph. Draw $P_{n+1}RS$ and P_1T parallel to Qn, and thus obtain the rectangle P_1RST.

Then $P_1R = ar^n - a$ and $P_1T = \frac{1}{r-1}$, and therefore the area of P_1RST represents the sum of n terms.

The significance of the rectangle P_1TQ1 will be seen immediately.

The same construction holds good for a G.P., the terms of which are decreasing.

Fig. 7 illustrates such a series.

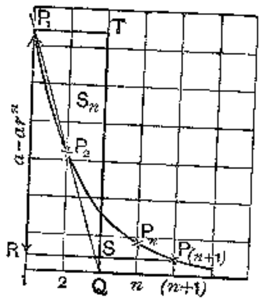


Fig. 7

The common ratio is, of course, less than unity.

The area P_1RST represents the sum of n terms.

As the number of terms increases, the last term approaches more and more the value 0, the graph gets closer and closer to the axis of n, and the line RS gets nearer and nearer to 1Q, the rectangle P_1RST becoming more nearly equal to the rectangle P_11QT. When the number of terms is infinite, the rectangle representing their sum differs by no measurable amount from the rectangle P_11QT.

Hence P_11QT represents the sum of an infinite number of terms of a G.P., the first term of which is a and the common ratio r, r being less than unity.

Now
$$\text{Area } P_11QT = 1P_1 \times 1Q$$
$$= a \times \frac{1}{1-r}$$
$$= \frac{a}{1-r},$$
$$\text{i.e.} \quad S_\infty = \frac{a}{1-r}.$$

Thus in fig. 7, the rectangle P_11QT represents the sum of an infinite number of terms in G.P., the greatest of which is $1P_1$.

The result, $S_\infty = \frac{a}{1-r}$, may be deduced from the relation, $S_n = \frac{a - ar^n}{1-r}$.

When r is less than unity, r^n becomes smaller and smaller as n increases.

The term ar^n can be made to differ from 0 by as small a quantity as we please, by making n large enough.

To make ar^n actually 0, n must be infinite.

In this case, $S_\infty = \frac{a - 0}{1 - r} = \frac{a}{1 - r}$.

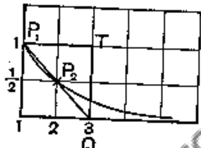


Fig. 8

EXAMPLE.—Find the sum of the series, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., without limit. Here $r = \frac{1}{2}$, and $S_\infty = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$.

Fig. 8 shows how the result can be obtained graphically.

Recurring Decimals.

A recurring decimal is an example of a G.P. with an infinite number of decreasing terms.

Thus: $\dot{\cdot 3} = \cdot 33333333333\ldots$
$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \text{etc.}$

The common ratio is $\frac{1}{10}$, and the sum of the terms is the value of the given decimal.

$$S_\infty = \frac{a}{1 - r} = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{10} \times \frac{10}{9} = \frac{3}{9} \text{ or } \frac{1}{3},$$

$$\text{i.e. } \cdot\dot{3} = \frac{3}{9} \text{ or } \frac{1}{3}.$$

Similarly, $\cdot 2\dot{3} = \frac{2}{10} + \left[\frac{3}{100} + \frac{3}{1000} + \ldots \text{etc.}\right]$

$$= \frac{2}{10} + \frac{\frac{3}{100}}{1 - \frac{1}{10}}$$

$$= \frac{2}{10} + \frac{3}{90}$$

$$= \frac{18 + 3}{90} \text{ or } \frac{{}^*2(10 - 1) + 3}{90} = \frac{20 - 2 + 3}{90}$$

$$= \frac{21}{90} \qquad\qquad = \frac{23 - 2}{90}$$

$$= \frac{7}{30}.$$

* The alternative steps explain the rule for converting a recurring decimal into a vulgar fraction.

Exercise XXVI (d)

1. Find the 8th term of the series 2, 6, 18, 54, etc.
2. Find the 9th term of the series -2, 6, -18, 54, etc.
3. Show that the series obtained by adding each term of a G.P. to the succeeding term is a G.P.
4. Show that the series obtained by subtracting each term of a G.P. from the succeeding term is a G.P.
5. Show that the series obtained by multiplying each term of a G.P. by the succeeding term is a G.P.
6. Insert three G.M.s between -3 and -768.
7. Find a straight line c, such that the side of the square of area 2·25 sq. in., the diagonal and c are in G.P.
8. Find the sum of the first six terms of the series 2, 6, 18, 54, etc., and of the series 2, -6, 18, -54, etc.
9. Find the sum of eight terms of the series 6, 3, $1\frac{1}{2}$, $\frac{3}{4}$, etc.
10. Find the sum of $\frac{3}{4}, -\frac{1}{2}, \frac{1}{3}, -\frac{2}{9} \ldots$ to 10 terms.
11. The fourth term of a G.P. is 24, and the ninth term -768. Find the eleventh term.
12. If s is the sum of a G.P., in which the first term is a and the last b, show that the common ratio is $\dfrac{s-a}{s-b}$.
13. Express as vulgar fractions, $\cdot 5\dot{2}\dot{4}$ and $\cdot 2\dot{3}0\dot{7}$, from first principles.
14. Find the sum of an infinite number of terms of the series $1 - \delta + \delta^2 - \delta^3 +$ etc., when δ represents a fraction.
15. Draw a straight line 2 in. long, and by successive bisection illustrate that the sum of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., without limit, is 2.

5. Harmonical Progression.

1. Numbers are in harmonical progression when their reciprocals are in arithmetical progression.

Examples.

(i) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, etc., are in H.P., because their reciprocals, 2, 3, 4, 5, etc., are in A.P.

(ii) $\frac{1}{2}, \frac{4}{7}, \frac{2}{3}, \frac{4}{5}, 1, 1\frac{1}{3}, 2$, etc., are in H.P., because their reciprocals, 2, $1\frac{3}{4}$, $1\frac{1}{2}$, $1\frac{1}{4}$, 1, $\frac{3}{4}$, $\frac{1}{2}$, etc., are in A.P.

(iii) Since the general form of the A.P. is

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots\ a+\overline{n-1}d,$$

the general form of the H.P. is

$$\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \frac{1}{a+3d},\ \ldots\ \frac{1}{a+\overline{n-1}d}.$$

2. If a, b and c are in H.P., then $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are in A.P.

Hence $$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b},$$

from which $$\frac{a-b}{ab}=\frac{b-c}{bc}$$

and $$\frac{a-b}{b-c}=\frac{a}{c},$$

i.e. the ratio of the excess of the first over the second to the excess of the second over the third, is equal to the ratio of the first to the third.

Harmonical progression is often defined in this way. The definition given in 1 is more easily remembered.

3. Problems on harmonical progressions are most conveniently solved by transforming the series by inversion into the corresponding arithmetical progression.

EXAMPLE.—Find the harmonic mean (H.M.) between a and b.

Let x be the H.M.

Then $\frac{1}{a}$, $\frac{1}{x}$ and $\frac{1}{b}$ are in A.P.

Hence $$\frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x},$$

$$\frac{2}{x}=\frac{1}{b}+\frac{1}{a},$$

$$\frac{2}{x}=\frac{a+b}{ab},$$

$$x=\frac{2ab}{a+b}.$$

4. There is no general formula for the sum of the terms of a harmonical progression.

Exercise XXVI (e)

1. Extend each of the series given in **5**, 1, by three terms.
2. Interpret the result of the example given in **5**, 3.
3. Insert two harmonic means between 3 and 12.
4. Show that a, b and c are in

$$\text{(i) A.P. if } \frac{a-b}{b-c} = \frac{a}{a},$$

$$\text{(ii) G.P. if } \frac{a-b}{b-c} = \frac{a}{b},$$

$$\text{(iii) H.P. if } \frac{a-b}{b-c} = \frac{a}{c}.$$

5. Construct and examine the graph of a harmonical progression.

6. Compound Interest.

In compound interest, at the end of a stated period, usually a year, the interest for that period is added to the principal, thereby giving a larger principal for the next period.

If interest is paid at the rate of, say, 5% per annum, then a principal P, invested for one year, gains an interest of ·05P, and amounts therefore to 1·05P, i.e. 1·05 times the principal at the beginning of the year. If this is allowed to remain for another year, at the end of this second year the sum amounts to

$$1{\cdot}05P \times 1{\cdot}05, \quad \text{i.e. to } P \times (1{\cdot}05)^2,$$

and so on. Thus the amounts at the end of successive years are as follows:

Year	1	2	3	....	n
Amount	$P(1{\cdot}05)$	$P(1{\cdot}05)^2$	$P(1{\cdot}05)^3$	....	$P(1{\cdot}05)^n$

If r is the rate per cent per annum, the amount at the end of the nth year is $P\left(1 + \frac{r}{100}\right)^n$

Notice that $\left(1 + \frac{r}{100}\right)$ is the sum that £1 amounts to in one year. If we call this amount a, the amount of P in n years becomes Pa^n, and the compound interest is therefore $Pa^n - P$.

Example.—Find the sum to which £120 amounts in 5 years at 4% per annum, compound interest.

$$\text{The amount (A)} = \text{P}\left(1 + \frac{r}{100}\right)^5$$

$$= 120 \times (1{\cdot}04)^5.$$

This is best evaluated by logarithms. Thus:

$$\log \text{A} = \log 120 + 5 \log 1{\cdot}04,$$

from which, $\text{A} = £146.$

7. Instalments.

Houses are frequently purchased through Building Societies by equal instalments extending over, say, 20 years. To find the annual payment per £100, reckoning compound interest at, say, 5% per annum.

Let £x = the annual payment, the first payment being made after 1 year. When the 20th payment is made, the first payment has borne interest for 19 years, and so on.

Value of 1st payment in 19 years = £ $x(1{\cdot}05)^{19}$.
,, 2nd ,, 18 years = $x(1{\cdot}05)^{18}$.
etc.
,, 20th ,, 0 years = x.

Value of all the payments

$$= x(1 + 1{\cdot}05 + \ldots + 1{\cdot}05^{19})$$

G.P. of common ratio 1·05

$$= \frac{x(1{\cdot}05^{20} - 1)}{1{\cdot}05 - 1}; \quad \text{(i)}$$

and this should equal the money to which £100 would amount in 20 years at the same rate, namely £100$(1{\cdot}05)^{20}$. (ii)

From equating (i) and (ii),

$$x = 100 \times (1{\cdot}05)^{20} \times \frac{1{\cdot}05 - 1}{1{\cdot}05^{20} - 1}$$

$$= £8{\cdot}021 = £8,\ 0s.\ 5d.$$

EXERCISE XXVI (F)

1. Find the compound interest on £250 for 3 years at 5% per annum.

2. Construct graphs to contrast the simple and compound interest on, say, £100 for various periods at, say, 4% per annum.

3. What sum will amount to £300 in 5 years at $2\frac{1}{2}$% per annum?

4. The population of a town in 1880 was 150,000. It increased by 5% each decade (10 years). What was the population in 1910?

5. The pressure in the bell-jar of an air-pump at the end of successive strokes was as follows:

Stroke - - - -	1st	2nd	3rd	4th	etc.
Pressure (lb. per sq. in.)	15	13·5	12·15	10·935	etc.

Calculate the pressure at the end of the 20th stroke.

6. The resistance in a motor armature circuit when the starter is on the various studs is given by the series:

$$R, \quad Rf, \quad Rf^2, \quad Rf^3 \ldots \text{etc.}$$

Calculate the resistance for 5 studs when R is 2 units and Rf^4 is 20 units.

7. The following are successive swings of a pendulum:

50, 49, 48·02, 47·06, etc., cm.

Calculate the length of the 10th swing.

8. In the "Achilles and Tortoise" race, if Achilles runs ten times as quickly as the tortoise, and the tortoise has 100 yd. start, then when Achilles has covered 100 yd. the tortoise has moved forward 10 yd., and so on.

Achilles has thus to cover a distance

$$(100 + 10 + 1 + \tfrac{1}{10} + \tfrac{1}{100} + \text{etc.})$$

before he catches up the tortoise, i.e. before the distance between him and the tortoise is 0. Find the distance by summing the series.

9. Find the annual payment to repay a loan of £500 in 10 years at 4% per annum.

8. Application of Logarithms to Geometrical Progressions.

Let the series be $a, ar, ar^2, \dots ar^{n-1}$.

The logs of the terms are:

$$\log a,\ (\log a + \log r),\ (\log a + 2\log r), \dots (\log a + \overline{n-1}\log r).$$

Notice that these terms have a common difference, namely, $\log r$. The terms are, therefore, in Arithmetical Progression.

The logarithms of the terms of a geometrical progression are in arithmetical progression, the common difference being the logarithm of the common ratio.

This result is of importance in science.

If the common ratio is less than unity, the common difference of the A.P. is negative.

The relation between a G.P. and the corresponding A.P. of the logs of its terms is illustrated graphically in fig. 9.

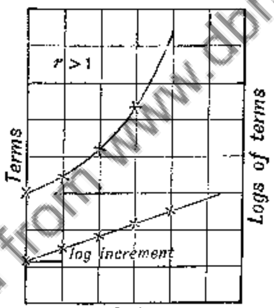


Fig. 9

The common difference between the logs of the terms is called the **Logarithmic Increment** if r is greater than unity, and the **Logarithmic Decrement** if less than unity.

EXERCISE XXVI (G)

1. Plot the logarithms of the terms of the following series, and state the logarithmic increment or decrement in each case:

(i) 1, 10, 100, 1000, etc. (ii) 100, 10, 1, 0·1, 0·01, 0·001, etc.

(iii) 2, 10, 50, 250, etc. (iv) 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

2. The following are the angular displacements of a pendulum, from its position of rest, during successive swings:

Right	10		9		8·2		7·4	degrees
Left		9·5		8·6		7·8		degrees

Find (*a*) the average logarithmic decrement (i) per half swing, (ii) per full swing;

(*b*) the angular displacement during its 10th excursion towards the left.

3. The heights to which a ball rises in successive rebounds are as follows:

$$3 \text{ ft.}, \quad 2 \text{ ft.}, \quad 1 \text{ ft. } 4 \text{ in.}, \quad 10\tfrac{2}{3} \text{ in.}$$

What relation exists between these numbers?

4. Compare the second term of the series:

$$1, \quad \tfrac{1}{2}, \quad \tfrac{1}{4}, \quad \tfrac{1}{8}, \text{ etc.},$$

with the sum of an infinite number of the terms following it.

5. Find the sum of varying numbers of terms of the series:

$$1, \quad \tfrac{1}{3}, \quad \tfrac{1}{9}, \quad \tfrac{1}{27}, \quad \tfrac{1}{81}, \text{ etc.},$$

and plot the results against the number of terms.

After examining the graph, say to what value the sum tends.

6. Find the value of the series:

$$1 - x + x^2 - x^3 + \text{etc.}, \quad \text{when } x < 1,$$

and show that it is the difference between the values of an infinite number of terms of each of the series:

$$1 + x^2 + x^4 + x^6 + \text{etc.} \quad \text{and} \quad x + x^3 + x^5 + \text{etc.}$$

7. The following numbers show the population of England and Wales for the years given.

Test whether they follow approximately the geometric law.

Year - - -	1851	1861	1871	1881	1891	1901	1911
Pop. (millions)	17·93	20·07	22·71	25·97	29·00	32·53	36·07

8. There is an important series:

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{(n-1)!} + \text{etc.},$$

where $3! = 1 \times 2 \times 3$, $4! = 1 \times 2 \times 3 \times 4$, etc.

Compare each of the terms after the first with the corresponding terms of the series:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}},$$

e.g. compare $\frac{1}{3!}$ with $\frac{1}{2^2}$.

Hence deduce that the sum of an infinite number of terms of the first series is less than 3.

9. Take the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} +$ etc., and find the ratio of the $(n + 1)$th term to the nth term. How does this ratio change when n is increased indefinitely, x remaining constant? After what value of n will the terms decrease successively?

10. The following numbers are taken from a table, showing:

(1) The annuity £100 will purchase.

(2) The price of an annuity of £10.

Age	50	52	54	56	58	60	62	64
(1) £	6/6/9	6/11/6	6/17/1	7/3/4	7/10/7	7/19/0	8/8/11	9/0/6
(2) £	157/16	152/0	145/18	139/10	132/19	125/15	118/8	110/16

Plot each, (1) and (2), against age, and by interpolation find the figures for age 57, and by extrapolation the probable figures for ages 45 and 68.

Draw other conclusions, if possible.

11. Prove that the product of the sum of, and the difference between, consecutive terms of a G.P. form another G.P.

12. Find the average of n terms of a G.P., the first term of which is a and the common ratio r.

13. Find the average of n terms of an A.P., the first term of which is a and the common difference d.

14. Find the sum of x terms of the series $1 + \frac{1}{3} + \frac{1}{9} +$ etc.
By how much does the sum to infinity exceed the sum of x terms?

15. Determine the number of years in which a sum of money will double itself at 5 per cent per annum compound interest.

16. The sum of the following series can be found by the same method as that for finding the sum of a geometric series. The expression obtained on subtracting contains a G.P.:

$$1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{\overline{n-1}}.$$

Find an expression for the sum of n terms.
Hence find the sum of ten terms of the following series:

$$1 + 6 + 27 + 108 + 405 + \text{etc.}$$

17. Write down the general term of the series:

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \text{etc.}$$

Find the ratio of the nth term to the preceding term. Under what conditions will the ratio be less than unity?

9. Permutations and Combinations.

The subject of Permutations and Combinations is concerned with the number of ways in which a number of things can be grouped together, arranged, or selected from a given set of things.

The distinction between permutations and combinations is as follows:

In **Permutations**, a different order of the same things is regarded as a different arrangement; whereas, in **Combinations**, no regard is paid to order, but only to the things which constitute a group.

For example, if we had two counters, one red and the other blue, then, placing them together, say one on top of the other, only one combination is possible, the position of the red counter with respect to the blue being of no consequence. On the other hand, there are two permutations possible, namely, one in which the red counter is above the blue, and the other in which the blue is above the red.

10. Permutations.

(1) To find the number of permutations of n things, taken r at a time.

This number is usually denoted by ${}_nP_r$.

The n different things are conveniently represented by the letters of the alphabet, but without restriction as to number.

Let a, b, c, d, etc., represent some of the different things. Consider the selection of:

(i) *One letter.*

Since there are n different letters, the number of ways in which one can be selected is n;

$$\text{i.e.} \quad {}_nP_1 = n.$$

(ii) *Two letters.*

One letter can be selected in n ways. Consider only one of these selections, say the letter a. There are now $(n - 1)$ letters left. A letter to be placed with a can be chosen from the remaining $(n - 1)$ letters in $(n - 1)$ ways. This is true for each of the n letters selected first, and, therefore, for all the n letters selected first there are $n(n - 1)$ ways of selecting the second, and, therefore, $n(n - 1)$ ways of selecting two things from n;

$$\text{i.e.} \quad {}_nP_2 = n(n - 1).$$

Note.—In these permutations, any two letters, say a and b, will occur twice; once when a was first selected and b from the remaining $(n - 1)$ letters, and once when b was first selected and a from the remaining $(n - 1)$ letters.

(iii) *Three letters.*

Two letters can be selected in $n(n - 1)$ ways. After each selection of two, $(n - 2)$ letters will remain. Another letter can be selected from the $(n - 2)$ letters in $(n - 2)$ ways, and since this is true for every one of the $n(n - 1)$ selections of two letters, three letters can be chosen in $n(n - 1)(n - 2)$ ways;

$$\text{i.e.} \quad {}_nP_3 = n(n - 1)(n - 2).$$

In this case, the same letters, say a, b, c, will occur no less than 6 times, the groups being as follows:

abc, *acb*, *bac*, *bca*, *cab*, *cba*.

(iv) Similarly,

$$\begin{aligned} {}_nP_4 &= n(n - 1)(n - 2)(n - 3), \\ {}_nP_5 &= n(n - 1)(n - 2)(n - 3)(n - 4), \\ &\text{etc.} \end{aligned}$$

Observe that in each case the last bracket consists of n minus one less than the number of letters taken at a time.

Hence, when r letters are taken at a time, the expression

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$$\text{i.e.} \quad {}_nP_1 = n.$$

(ii) *Two letters.*

One letter can be selected in n ways. Consider only one of these selections, say the letter a. There are now $(n - 1)$ letters left. A letter to be placed with a can be chosen from the remaining $(n - 1)$ letters in $(n - 1)$ ways. This is true for each of the n letters selected first, and, therefore, for all the n letters selected first there are $n(n - 1)$ ways of selecting the second, and, therefore, $n(n - 1)$ ways of selecting two things from n;

$$\text{i.e.} \quad {}_nP_2 = n(n - 1).$$

Note.—In these permutations, any two letters, say a and b, will occur twice; once when a was first selected and b from the remaining $(n - 1)$ letters, and once when b was first selected and a from the remaining $(n - 1)$ letters.

(iii) *Three letters.*

Two letters can be selected in $n(n - 1)$ ways. After each selection of two, $(n - 2)$ letters will remain. Another letter can be selected from the $(n - 2)$ letters in $(n - 2)$ ways, and since this is true for every one of the $n(n - 1)$ selections of two letters, three letters can be chosen in $n(n - 1)(n - 2)$ ways;

$$\text{i.e.} \quad {}_nP_3 = n(n - 1)(n - 2).$$

In this case, the same letters, say a, b, c, will occur no less than 6 times, the groups being as follows:

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

(iv) Similarly, ${}_nP_4 = n(n - 1)(n - 2)(n - 3)$,

${}_nP_5 = n(n - 1)(n - 2)(n - 3)(n - 4)$,

etc.

Observe that in each case the last bracket consists of n minus one less than the number of letters taken at a time.

Hence, when r letters are taken at a time, the expression representing ${}_nP_r$ will consist of a similar product, the last bracket of which will contain n minus $\overline{r-1}$;

i.e. ${}_nP_r = n(n-1)(n-2)(n-3)(n-4)\dots(n-\overline{r-1})$;

i.e. ${}_nP_r = n(n-1)(n-2)(n-3)(n-4)\dots(n-r+1)$. (i)

(2) *Special case.*

To find the number of permutations of n things taken n at a time.

In this case $r = n$.

Hence
$$ {}_nP_n = n(n-1)(n-2)(n-3)\dots(n-n+1), $$
$$ {}_nP_n = n(n-1)(n-2)(n-3)\dots(1). $$

This product, the factors of which run down to 1, is called *Factorial* n, and is written as either $\underline{|n}$ or $n!$,

i.e. ${}_nP_n = n!$. . . . . . . (ii)

(3) In order to extend product (i) to the factorial n, it would be necessary to multiply it by a product the factors of which range from $(n-r)$, one less than $(n-r+1)$, to 1;

i.e. by $(n-r)!$.

It follows that
$$ {}_nP_r = \frac{n!}{(n-r)!}. \quad \dots\dots \quad \text{(iii)} $$

This is a more convenient form than that given in equation (i).

11. (1) To find the number of combinations of n things taken r at a time (${}_nC_r$).

The number of permutations is $\frac{n!}{(n-r)!}$.

Now, as stated in § **10** (i) (ii) and (iii), the same set of letters will occur in more than one permutation, but in different order.

In fact the number of times the same set of letters will occur is the number of permutations of r things taken r at a time, i.e. by 2 (ii), $r!$.

In combinations, these $r!$ arrangements count as one combination only, and as this is true for all such sets in the permutations of n things, r at a time, we have:

$$ {}_nC_r = \frac{{}_nP_r}{r!}, \quad \text{i.e. } {}_nC_r = \frac{n!}{(n-r)!\,r!}; $$

(i) e.g. ${}_nC_3 = \frac{{}_nP_3}{3!} = \frac{{}_nP_3}{3\times 2\times 1} = \frac{{}_nP_3}{6}$.

(ii) $$_{10}C_4 = \frac{10!}{6! \times 4!} = \frac{10.9.8.7}{4.3.2.1} = 210.$$

(2) To show that $_nC_r = {_nC_{n-r}}$.

The equality is readily established by the following consideration.

For every combination of r things taken from the n things, a combination of $(n - r)$ things is left.

$$\therefore \; _nC_r = {_nC_{n-r}}.$$

The equality also follows from the fact that each equals

$$\frac{n!}{r!(n-r)!}.$$

(3) The following example shows the application of combinations to problems on probability or chance.

If 52 cards are dealt to 4 players, what is the chance that a player will receive 4 aces?

13 cards can be dealt in $_{52}C_{13}$ ways. Of these, the number of times the 4 aces and 9 other cards are dealt, is $_{48}C_9$.

Hence the chance of one player receiving 4 aces is $\frac{_{48}C_9}{_{52}C_{13}}$,

$$\text{i.e. } \frac{48!}{39!\,9!} \div \frac{52!}{39!\,13!} = \frac{11}{4165}, \text{ i.e. } 11 \text{ in } 4165.$$

Exercise XXVI (h)

1. Calculate: $_8P_8$, $_5P_3$, $\frac{_8P_2}{_8P_4}$, $\frac{_{10}P_6}{_6P_2}$, $_5C_3$, $_6C_5$, $\frac{_6C_3}{_6C_2}$, $\frac{_5P_2}{_5C_2}$.
2. There are 20 competitors for a first, second or third prize. In how many ways may the prizes be won?
3. How many different amounts can be paid out of a till containing a sovereign, half-sovereign, crown, half-crown, florin, shilling, sixpence, three-penny piece, penny, and a half-penny, if three coins only are to be paid out?
4. In how many ways could 5 playing cards be dealt from a pack of 52? How many sets would contain cards of particular different numbers, independent of suit?
5. How many words of three letters can be formed from 6 consonants and 4 vowels, each word to contain one vowel?

6. In how many different orders could 5 men be seated round a circular table? (Place one man, then select the orders of the remaining four. If clockwise and anti-clockwise arrangements are considered alike, halve the result.)

7. How many integers of (*a*) 3 digits, (*b*) not more than 3 digits can be formed with the figures (i) 1 to 5, (ii) 0 to 5, each figure being used only once in each number?

8. Show that the number of permutations of 5 things taken 3 at a time when each may occur any number of times is 5^3.
   Find the answers to Ex. 7 when each figure may occur any number of times.

9. Show that the total number of combinations of n different things taken any number at a time is $2^n - 1$. How many products can be formed from the letters a, b, c, d?

12. The Binomial Theorem.*

(1) To find $(a + b)^n$.

That is, to find the product of n factors, each of which is $(a + b)$. This is readily deduced from the theory of combinations.

(i) Clearly, the first terms is a^n, and the last b^n.

(ii) The second is of the kind, $a^{n-1}b$.

The term is formed by selecting one b from the n b's available and placing it with the product of the $(n - 1)$ a's of the brackets from which the b is not chosen.

Since the b can be chosen in n ways, there are n such products, and the coefficient of the term is therefore n.

The term is thus, $na^{n-1}b$.

(iii) The third term is of the kind $a^{n-2}b^2$.

Two b's can be selected from the n b's in ${}_nC_2$ ways.
(It does not matter in what order they are chosen.)
The third term is, therefore, ${}_nC_2\, a^{n-2}b^2$.

(iv) Similarly, the coefficients of the terms following are ${}_nC_3$, ${}_nC_4$, ${}_nC_5$, etc., the last being ${}_nC_n$, which is, of course, 1; this agrees with the statement in (i).

* Formulated by Newton.

(v) Since ${}_nC_1 = n$, we can write the expansion in the form:

$$(a + b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + {}_nC_3 a^{n-3}b^3 + \dots$$
$$\dots + {}_nC_r a^{n-r}b^r + \dots + {}_nC_n b^n$$

$$= a^n + na^{n-1}b + \frac{n!}{(n-2)!2!} a^{n-2}b^2 + \frac{n!}{(n-3)!3!} a^{n-3}b^3 + \dots$$
$$\dots + \frac{n!}{(n-r)!r!} a^{n-r}b^r + \dots + b^n$$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{1.2} a^{n-2}b^2$$
$$+ \frac{n(n-1)(n-2)}{1.2.3} a^{n-3}b^3 + \dots + b^n.$$

Observe that $\frac{n!}{(n-r)!r!} a^{n-r}b^r$ is the $(r+1)$th term.

(2) The following are important properties of the terms of the expansion of $(a + b)^n$.

(i) The coefficients of terms equidistant from the beginning and the end are equal.

Thus, ${}_nC_2$ and ${}_nC_{n-2}$ are the coefficients of the terms $a^{n-2}b^2$ and a^2b^{n-2}, which are equidistant from the beginning and the end, and by § **11** (2),

$${}_nC_2 = {}_nC_{n-2}.$$

(ii) The sum of the coefficients is 2^n.

This is found by putting a and b each equal to 1.

(iii) The greatest *coefficient.*

The coefficient of the rth term is ${}_nC_{r-1}$,

,, ,, $(r + 1)$th term, ${}_nC_r$,

,, ,, $(r + 2)$th term, ${}_nC_{r+1}$.

For the coefficient of the $(r + 1)$th term to be the greatest, we must have:

$$\text{(i) } \frac{{}_nC_r}{{}_nC_{r-1}} > 1 \quad \text{and} \quad \text{(ii) } \frac{{}_nC_r}{{}_nC_{r+1}} > 1.$$

From (i), $\quad \frac{n-r+1}{r} > 1$, and $\therefore \frac{n+1}{2} > r$.

From (ii), $\quad \frac{r+1}{n-r} > 1$, and $\therefore r > \frac{n-1}{2}$.

That is, for the coefficient of the $(r + 1)$th term to be the greatest, r must be greater than $\frac{n-1}{2}$ and less than $\frac{n+1}{2}$; e.g. if n is 8, r must be greater than $3\frac{1}{2}$ and less than $4\frac{1}{2}$.

Hence, r is 4 and the $(r + 1)$th term the 5th.

The actual coefficient is ${}_8C_4$, i.e. 70.

Exercise.—Verify this by calculating the neighbouring coefficients.

If n is 7, r must be greater than 3 and less than 4; but r must be an integer, and it will be found that both values satisfy the conditions. That is, the 4th and 5th terms have equal coefficients, and the greatest.

(3) The expansion of $(a - b)^n$ is obtained by substituting $-b$ for b in the expansion of $(a + b)^n$. It follows that the two expansions differ in signs only. In the expansion of $(a - b)^n$, the signs run alternately plus and minus.

Exercise XXVI (i)

1. Write down the expansions of $(1 + x)^n$ and $(1 - x)^n$.
2. Find the coefficient of x^5 in the expansion of $(2 - x)^8$.
3. Find the sum of the coefficients in the expansion of $(a + x)^6$.
4. Find the sum of the coefficients in the expanded forms of $(2 + x)^5$ and $(2 - x)^5$.
5. How many terms are there in the expansion of $(a + b)^9$? Which term has the greatest coefficient, and what is its value?
6. Expand $(1 + x)^{10}$, and find the greatest term when $x = 2$.

Revision Exercise III

1. Plot the graph of $y = (x-2)(x+3)$ between $x = +3$ and $y = -4$, and use the graph to find approximately the roots of the equation

$$4x^2 + 4x - 11 = 0.$$

State your construction, and give reasons for your inferences.

2. Plot the point $x = 3$, $y = 4$, and the straight line $y = 2$. Draw the locus of a point which moves so that its distance from the point 3, 4, is equal to its perpendicular distance from the straight line $y = 2$.

Verify that the equation to the locus is $y = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{21}{4}$.

3. A peg top has the form of an equilateral cone and a hemisphere placed base to base. If the diameter of the base of the cone is s, find the volume of the peg top.

4. Show that $\frac{\pi}{2}-\cos^{-1}\frac{x}{a}=\sin^{-1}\frac{x}{a}$.

5. Given that $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k,$$

show that (i) $\frac{a+c+e}{b+d+f}=\frac{a}{b}$;

(ii) $\frac{ab+cd+ef}{b^2+d^2+f^2}=\frac{a^2+c^2+e^2}{ab+cd+ef}=\frac{a}{b}$.

6. A passenger in a train travelling at 60 miles per hour observes that it takes 8 sec. to pass a train 110 yd. long, going in the same direction. How long would it have taken if the trains had been travelling in opposite directions?

If the first train were 132 yd. long, in what time would they pass one another in each case?

7. A man puts by £100 at the beginning of each year to accumulate at compound interest at 4 per cent per annum. Show that when he has put by his tenth instalment, he has accumulated a fund of £1200. [Assume $(1{\cdot}04)^{10}=1{\cdot}48$.]

8. The first two terms of an A.P. are $\frac{1}{\sqrt{2}}$ and $\frac{1}{1+\sqrt{2}}$; find the third term.

What would the third term be if the numbers were in G.P.?

9. If $\sin\theta=\frac{x}{a}$, show that $\cos\theta=\frac{\sqrt{a^2-x^2}}{a}$ and $\tan\theta=\frac{x}{\sqrt{a^2-x^2}}$.

10. Find how far a sphere of diameter 2″ will sink into a conical wine-glass, of which the depth and the diameter of the mouth are each $2\frac{1}{2}$″.

Find also what volume of the sphere is inside the wine-glass.

CHAPTER XXVII

AN INTRODUCTION TO THE DIFFERENTIAL AND INTEGRAL CALCULUS *

1. Rate of Change of Simple Functions.

Consider first a linear function, i.e. a function the graph of which is a straight line.

Let y be a linear function of x, the general relation being $y = ax + b$, a and b being constants.

* The pioneers in this branch of mathematics were Newton (1642–1727) and Leibniz (1646–1716).

Let x_1 and y_1, and x_2 and y_2, respectively, be simultaneous values of x and y (fig. 1).

Then y changes from y_1 to y_2, when x changes from x_1 to x_2. The change in $x = x_2 - x_1$, and the change in $y = y_2 - y_1$.

The ratio of the change in y to the change in x is, $\frac{y_2 - y_1}{x_2 - x_1}$.

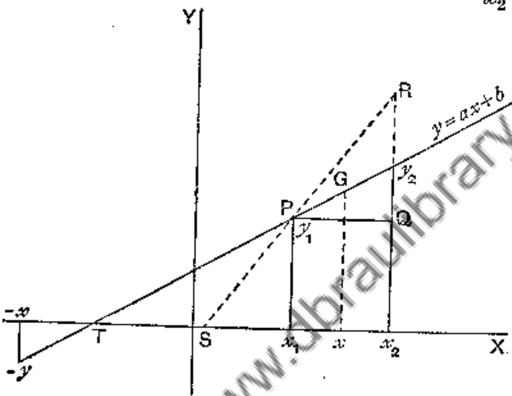

Fig. 1

It is already known that, in the case of the straight-line graph, this ratio is constant for all values of x_1 and x_2. Moreover, since Δy_2QP is similar to ΔPx_1T,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{x_1P}{x_1T},$$

which is the tangent of the angle x_1TP, i.e. of the angle the graph makes with the axis of x, and which we have seen is a measure of the gradient of the graph.

In this case, the tangent is positive, and this is true even when the graph is produced below the axis of x, for then both x and y are negative, and the quotient therefore positive.

We may say that the quotient $\frac{y_2 - y_1}{x_2 - x_1}$ represents the gradient at the point G, and has been determined by taking two values of x, one on each side of that for G, and finding the quotient,

$$\frac{\text{Change in } y}{\text{Change in } x}.$$

When these changes, and therefore the sides of the ΔPQy_2, are small, it is easier to find the quotient from ΔPx_1T.

This quotient represents the **rate** at which y, i.e. $ax + b$, changes as x is changed, and, in order to understand the full significance of this, compare two linear graphs passing through P.

In fig. 1, it is seen that, for the graph PR, *the change in* y, *viz.* QR, *when the change in* x *is* $(x_2 - x_1)$, *is greater than that for the graph* TP, *viz.* Qy_2, *for the same change in* x. *The gradient of* PR *is steeper than that of* TP, *the difference being*

$$\frac{QR - Qy_2}{x_2 - x_1}.$$

This difference is equal to

$$\tan\angle QPR - \tan\angle QPy_2,$$

and since $\angle QPR = \angle XSP$, *and* $\angle QPy_2 = \angle XTP$,

to $\tan\angle XSP - \tan\angle XTP.$

By Geometry, extr. $\angle XSP$ *is greater than* opp. intr. $\angle XTP$.

Returning to the relation $y = ax + b$, we know from the chapter on the linear graph that the gradient is a.

The rate of change of y, i.e. $ax + b$, as x changes, is therefore a.

The points to remember are:

(i) In the straight-line graph the gradient is constant, and is measured by the tangent of the angle the graph makes with the axis of x.

(ii) The gradient of the graph measures the rate at which y changes with x, and in the straight-line graph this rate is constant.

2. When the changes are very small, it is usual to employ special symbols to represent them. Thus a small change in x is represented by δx, and the corresponding change in y by δy.

The Greek letter δ (d) *is not in this case a multiplier, but when placed before another letter, it must be taken to indicate a small change in the quantity represented by the letter. Referring to the figure, in such a case* PQ *represents* δx, *and* Qy_2 δy *and the triangle* PQy_2 *may be as small as we please.*

The quotient $\dfrac{\text{change in } y}{\text{small change in } x}$ is then written $\dfrac{\delta y}{\delta x}$.

If $y = ax + b$, we now know that $\dfrac{\delta y}{\delta x} = a$, the gradient of the straight-line graph.

EXERCISE XXVII (A)

1. Draw a graph representing the distance covered by a moving body, as shown in the following table:

Time (sec.)	0	1	2	3	4	5	6
Distance (ft.)	0	3	6	9	12	15	18

Find the rate of change of distance with time, at definite instants (say at $1\frac{1}{2}$, 2, $3\frac{1}{4}$, etc., seconds after time 0), by taking small intervals containing the instant and determining the quotient $\frac{\text{change in distance}}{\text{change in time}}$.

The quotient, as you know, is the velocity of the body. What do you know of the velocity in this case?

2. Draw the graphs of

(i) $2x + 5$, (ii) $2x$, (iii) $5 - 2x$,
(iv) $-2x$, (v) $\frac{1}{4}x - 2$, (vi) $\frac{1}{4}x + 2$,

and in each case determine at various points the rate at which the value of the expression changes with respect to x.

3. What is the gradient of the graph $y = 0x + b$?

What therefore is the rate of the change of y with respect to x?

4. Take the function $y = 2x + 3$, and find the value of y when x is, say, 3. Then increase the value of x to, say, 3·1, and find the new value of y.

Now calculate the quotient $\frac{\text{change in } y}{\text{change in } x}$.

Repeat the process with x equal to, say, -4 and $-4{\cdot}05$, or $-3{\cdot}95$.

Compare the quotients.

Repeat this exercise with other linear functions of x.

3. Rate of Change of Non-Linear Functions.

When we consider functions which are not linear, the determination of the rate of change of the function is not so easy.

The more important functions, other than the linear function, of which we have drawn graphs, are:

(i) $y = ax^2$ (p. 202), (ii) $y = ax^2 + c$ (p. 203),
(iii) $y = ax^2 + bx + c$ (p. 209), (iv) $x^2 + y^2 = a^2$ (p. 226),
(v) $y = \frac{1}{x} + c$ (p. 155), and
(vi) the trigonometrical functions (p. 269).

The graphs of these are curves.

4. Limiting Value of Ratio of Changes. Gradient of Tangent.

Consider the graph of $y = x^2$ and let us find the rate of change of y with x at the point for which x is, say, 3 and y therefore 9. If we change the value of x to 3·5,

$$y \text{ becomes } (3{\cdot}5)^2 = 12{\cdot}25,$$

then $$\frac{\text{change in } y}{\text{change in } x} = \frac{12{\cdot}25 - 9}{3{\cdot}5 - 3} = 6{\cdot}5.$$

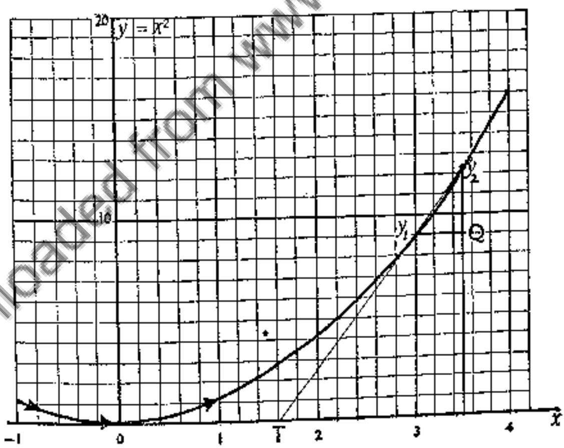

Fig. 2

If y_1 and y_2 are the points on the graph (fig 2) representing these values of x and y, then 6·5 is the gradient of the straight line joining y_2 and y_1.

Next, change x from 3 to 3·4 and y therefore from 9 to $(3{\cdot}4)^2$, i.e. to 11·56,

then $$\frac{\text{change in } y}{\text{change in } x} = \frac{11{\cdot}56 - 9}{3{\cdot}4 - 3} = 6{\cdot}4.$$

y_2 is now nearer to y_1 and the straight line joining the points has a gradient of 6·4 as compared with 6·5 previously. The line has altered its slope.

If these calculations are continued for changes in x of ·3, ·2, ·1, the quotients obtained are 6·3, 6·2, 6·1. Now if these are plotted as in fig. 3 against the changes in x and the points joined to make

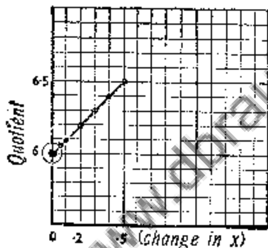

Fig. 3

a graph, then if the graph is extended backwards to give the value of the quotient for the change 0 in x, the result is 6. For this value 6, y_2 has reached y_1 in fig. 2, and straight line y_2y_1T has become a tangent to the curve at y_1. The gradient of this tangent is 6.

Thus, although the triangle y_1Qy_2 has disappeared, as well as the changes in x and y, yet the value towards which the quotient of the changes steadily approaches (or *tends*) can be found from the gradient of the tangent at the point representing the values of x and y.

The reader is reminded that a curve and its tangent have the same direction and therefore the same gradient at the point of contact.

These calculations should be made for changes at other values of x. At $x = 5$, for example, the quotient $\dfrac{\text{change in } y}{\text{change in } x}$ will be found to get nearer and nearer to 10, which we call the *limiting value*, or *limit*, of the quotient when the changes become smaller and smaller without limit.

This limiting value is the *exact* rate of change of y (i.e. x^2) with respect to x when x is equal to 5.

We have represented small changes in x and y by δx and δy, and $\frac{\delta y}{\delta x}$ is the corresponding quotient $\frac{\text{change in } y}{\text{change in } x}$. The values found for $\delta y/\delta x$, by making δx small, are approximations to the exact value. To distinguish the final or limiting value of the quotient, the lettering is changed from $\frac{\delta y}{\delta x}$ to $\frac{dy}{dx}$, which should be regarded as a single symbol for the quotient and not as consisting of two terms. It follows that for the graph of $y = x^2$, $\frac{dy}{dx}$ represents the gradient of the tangent to the curve at the point considered, e.g. at $x = 3$, $\frac{dy}{dx} = 6$.

5. Differentiation. Differential Coefficient.

All the foregoing can be expressed algebraically, thus:

(i) Find $\frac{dy}{dx}$ when $y = x^2$. Let x_1 and x_2 be near values of x and y_1 and y_2 the corresponding values of y.

Then $$y_2 = x_2^2 \text{ and } y_1 = x_1^2$$

and the quotient,

$$\frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x_2^2 - x_1^2}{x_2 - x_1} = x_2 + x_1.$$

Let the difference between x_2 and x_1 be diminished so that they both approach the value x.

Then $$x_2 + x_1 \text{ tends to } 2x,$$

$$\text{i.e. } \frac{dy}{dx} = 2x.$$

This agrees with the result found by graph, namely, at $x = 3$ the gradient of the tangent, which represented the rate of change, was found to be 6.

(ii) Another and a more general method is as follows:

Let x change by the small increment δx, and let δy be the resulting change in y.

Then, $$y + \delta y = (x + \delta x)^2;$$
$$\therefore\ \delta y = (x + \delta x)^2 - x^2$$
$$= (2x + \delta x)\delta x,$$
and $$\therefore\ \frac{\delta y}{\delta x} = 2x + \delta x.$$

Now let δx diminish until the limit $\delta x = 0$ is reached, then $\frac{\delta y}{\delta x}$ becomes $\frac{dy}{dx}$ and $2x + \delta x$ becomes $2x$,

$$\text{i.e. } \frac{dy}{dx} = 2x.$$

The operation is called DIFFERENTIATION; and the value of $\frac{dy}{dx}$ is called the **differential coefficient** or **derivative** of the function of x. Thus $2x$ is the differential coefficient of x^2. This may be written $\frac{d}{dx}(x^2) = 2x$.

Notice that for $x = -3$, $\frac{dy}{dx} = 2 \times -3 = -6$. This agrees with the direction of the graph in fig. 2, for at $x = -3$ the gradient is downwards, and therefore negative. The sign and value of the gradient should be examined along the curve from left to right. It will be seen that the gradient on the left is negative, but diminishes numerically until the lowest point is reached, where it is 0, then on the right becomes positive and increases numerically.

6. The Quotient $\frac{0}{0}$.

The symbol $0 \div 0$ is meaningless as it stands, since any number of times 0 is 0. However, if the noughts have been obtained by reducing the numerator and denominator of a fraction according to some relation between them until they are less than any assigned amount, the quotient generally approaches and finally reaches a definite value.

E.g., from § **5**, p. 297, it is seen that $\frac{\sin\delta}{\delta}$ gets nearer and nearer to 1 as δ and $\sin\delta$ get nearer and nearer to 0. We express this by saying that $\frac{\sin\delta}{\delta}$ tends to 1 as δ tends to 0, or that

$\frac{\sin\delta}{\delta} \to 1$ as $\delta \to 0$, or that the limit of $\frac{\sin\delta}{\delta}$, as δ tends to 0, is 1.

The fact that such quotients have a limiting value is the basis of the calculus.

EXERCISE XXVII (B)

1. Find the values of $\frac{dy}{dx}$ when $y = x^2$ for the following values of x, viz. $-6, -4, -1, 0, +1, +4, +6$.
2. What is the gradient of the graph of $y = x^2$ at its lowest point?
3. If $y = ax^2 + bx$, using the method of example (ii) (p. 333), show that $\delta y = a(2x + \delta x)\delta x + b\,\delta x$,

and that
$$\frac{dy}{dx} = 2ax + b = \frac{d(ax^2)}{dx} + \frac{d(bx)}{dx}.$$

7. General Formula for the Differential Coefficient of a Power of x.

To find $\frac{dy}{dx}$ when $y = x^n$.

(i) Let n be a positive integer.

Employing method (i) (p. 333), let x_1 and x_2 be near values of x and y_1 and y_2 the corresponding values of y.

Then,
$$y_2 = x_2^n,$$
$$y_1 = x_1^n.$$

By subtraction,
$$y_2 - y_1 = x_2^n - x_1^n,$$

and $\therefore$
$$\frac{\text{change in } y}{\text{change in } x} = \frac{x_2^n - x_1^n}{x_2 - x_1}$$
$$= x_2^{n-1} + x_2^{n-2}x_1 + x_2^{n-3}x_1^2 + \text{etc.} \ldots + x_1^{n-1}.$$

Let the difference between x_2 and x_1 diminish so that $x_2 \to x$, and $x_1 \to x$. Then each term on the right becomes x^{n-1}, and since there are n terms, the sum is nx^{n-1}.

$$\therefore \frac{\text{change in } y}{\text{change in } x} \to nx^{n-1},$$

$$\text{i.e.} \quad \frac{dy}{dx} = nx^{n-1}.$$

(ii) Let n be a positive fraction $\frac{p}{q}$, so that $y = x^{p/q}$, where p and q are positive integers.

Then $$y^q = x^p,$$
and, proceeding as in (i),
$$y_2^q - y_1^q = x_2^p - x_1^p,$$
which gives at once
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{x_2^{p-1} + x_2^{p-2}x_1 + \ldots + x_1^{p-1}}{y_2^{q-1} + y_2^{q-2}y_1 + \ldots + y_1^{q-1}}.$$
$$\therefore \quad \frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}}$$
$$= \frac{p}{q}\,\frac{x^{p-1}}{x^{p(q-1)/q}}$$
$$= \frac{p}{q}\,\frac{x^{p-1}}{x^{(p-p/q)}}$$
$$= \frac{p}{q}\,x^{(p/q)-1}$$
$$= nx^{n-1},$$
the same result as in (i).

(iii) Let n be a negative integer or fraction, say $n = -m$, where m is positive.

$$\therefore \quad y = \frac{1}{x^m},$$
or $$\frac{1}{y} = x^m;$$
$$\therefore \quad \frac{1}{y_2} - \frac{1}{y_1} = x_2^m - x_1^m,$$
$$-\frac{y_2 - y_1}{y_2 y_1} = (x_2 - x_1)(x_2^{m-1} + x_2^{m-2}x_1 + \ldots + x_1^{m-1}).$$
$$\therefore \quad \frac{y_2 - y_1}{x_2 - x_1} = -y_2 y_1(x_2^{m-1} + \ldots + x_1^{m-1}).$$
$$\therefore \quad \frac{dy}{dx} = -y^2(mx^{m-1})$$
$$= -\frac{mx^{m-1}}{x^{2m}}$$
$$= -mx^{-m-1}$$
$$= nx^{n-1},$$
as before.

Hence in all cases,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

If $y = ax^n$, where a is a constant, clearly the above process will give $\frac{dy}{dx} = anx^{n-1}$.

Example i.—If $y = 3x^2$, a being 3 and n, 2,

then $$\frac{dy}{dx} = 3\times 2x = 6x.$$

Example ii.—If $y = 3x^{-2} = \frac{3}{x^2}$,

then, since $n - 1 = -3$, $\frac{dy}{dx} = 3 \times -2x^{-3} = -\frac{6}{x^3}$.

Example iii.—If $y = 3\sqrt{x} = 3x^{\frac{1}{2}}$,

then $$\frac{dy}{dx} = 3 \times \tfrac{1}{2}x^{(\frac{1}{2}-1)} = \tfrac{3}{2}x^{-\frac{1}{2}} \text{ or } \frac{3}{2\sqrt{x}}.$$

8. Area. Integration.

Let CPQD (fig. 4) be the graph of a function $f(x)$;

$$OA = a,\ AC = f(a);\ OM = x,\ MP = f(x).$$

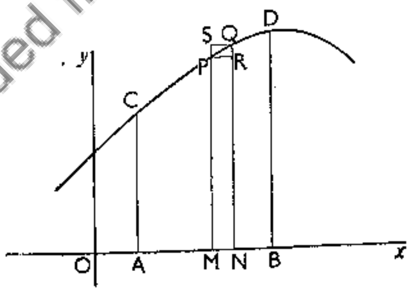

Fig. 4

Let AC be fixed, MP variable, and let A(x) be the measure of the area AMPC. A(x) is therefore a function of x, which has the value 0 when $x = a$.

We obtain the derivative $\frac{dA}{dx}$ in the usual way by finding the limit, when $\delta x \to 0$, of the fraction

$$\frac{A(x + \delta x) - A(x)}{\delta x}.$$

If $\delta x = MN$, $\quad A(x + \delta x) = \text{area ANQC}$, and

$$A(x + \delta x) - A(x) = \text{area MNQP}.$$

The latter area is the sum of the rectangle MNRP, of area $y\,\delta x$ or $f(x)\,\delta x$, and the small curvilinear triangle PRQ. But PRQ is less than the rectangle PRQS, the area of which is $\delta x \,.\, RQ$. Thus

$$\frac{A(x + \delta x) - A(x)}{\delta x} = \frac{f(x)\,\delta x + PRQ}{\delta x}$$

$$= f(x) + \frac{PRQ}{\delta x};$$

and $\frac{PRQ}{\delta x}$ is less than RQ, which $\to 0$ when $\delta x \to 0$.

$\therefore$ the limit of $\frac{A(x + \delta x) - A(x)}{\delta x}$ is $f(x)$; i.e.

$$\boldsymbol{\frac{dA}{dx} = f(x).}$$

Example i.—Find the area between the curve $y = x^2$, the axis of x, and the ordinates at $x = 1$, $x = 4$.

Here $$\frac{dA(x)}{dx} = x^2.$$

But $$\frac{d}{dx}(\tfrac{1}{3}x^3) = x^2.$$

$$\therefore \frac{d}{dx}\left\{A(x) - \tfrac{1}{3}x^3\right\} = 0;$$

$\therefore$ $A(x) - \frac{1}{3}x^3$ is constant, or

$$A(x) = \tfrac{1}{3}x^3 + C.$$

To find C, use the fact that $A(x) = 0$, when $x = 1$.

$$\therefore 0 = \tfrac{1}{3} + C, \text{ or } C = -\tfrac{1}{3},$$

$$\therefore A(x) = \tfrac{1}{3}x^3 - \tfrac{1}{3}.$$

This is the area A(x) from the ordinate at $x = 1$ to the ordinate at x. To find the area up to the ordinate at $x = 4$, put $x = 4$.

$$\therefore \text{ Area required} = \tfrac{1}{3} \times 4^3 - \tfrac{1}{3} = 21.$$

(You can now prove the result stated at the end of section 5, Chap. XXII.)

In the above process, we had to find a function which had a given derivative. The derivative of a function F(x) is usually denoted by F′(x), i.e. if

$$z = F(x),$$

then $$\frac{dz}{dx} = F'(x).$$

To find an area we have to find F(x) when we are given F′(x).

Example ii.—To find the area between the curve $y = f'(x)$, the axis of x, and the ordinates at $x = a$ and $x = b$.

Here $$\frac{dA(x)}{dx} = f'(x),$$

so that, as in Ex. i, $$A(x) = f(x) + C.$$

But $$A(a) = 0,$$

$$\therefore \quad 0 = f(a) + C,$$

$$A(x) = f(x) - f(a).$$

$$\text{Area required} = A(b)$$
$$= f(b) - f(a).$$

9. In fig. 4, clearly we might divide the whole area into little strips like MNQP. Suppose for simplicity that δx has the same value for each of those strips, say $\delta x = h$.

Each strip can be divided into a rectangle and a triangle, as in fig. 4. The sum of the triangular areas

$$= \tfrac{1}{2}h \times \text{sum of their heights}$$
$$= \tfrac{1}{2}h(BD - AC), \text{ which} \to 0 \text{ when } h \to 0.$$

The sum of the rectangles is

$$h\{f(a) + f(a+h) + f(a+2h) + \ldots + f(b-h)\}. \qquad \text{(I)}$$

Hence the limiting value of the sum (I), when $h \to 0$, is the area ABDC under the curve.

If then $F(x)$ is a function such that $f(x) = F'(x)$, we see, as in Ex. (ii) of last section, that the limit, when $h \to 0$, of the sum

$$\left.\begin{array}{c} h\{f(a)+f(a+h)+f(a+2h)+\ldots+f(b-h)\} \\ \text{is } F(b)-F(a). \end{array}\right\} \quad . \quad \text{(II)}$$

This is a very important result, which is used in all parts of mathematics, both pure and applied. (Compare Chap. XXII, section **1**.)

It will be noticed that in (II) the sum in { } becomes greater and greater without limit—or becomes infinite—when $h \to 0$. The product on the left of (II) has therefore the form $0 \times \infty$; and though this symbol is meaningless by itself, we see that it may have a definite finite value when it is connected with a limiting process. Compare section **6**.

We may write the result (II), just found, in the form:

$$\textbf{Limit when } \delta x \to 0 \textbf{ of } \sum_{x=a}^{x=b} f(x)\delta x = F(b) - (Fa).$$

The notation regularly used for the expression on the left is

$$\int_a^b f(x)\,dx.$$

The sign $\int$, originally a long *s*, is called the *integral* sign, and $\int_a^b f(x)\,dx$ is called **the definite integral** of $f(x)$ with respect to x, from a to b; a and b are called the *limits* of the integral.

We may also have an **indefinite integral** of $f(x)$ with respect to x, with no limits specified such as a, b. The indefinite integral is written simply

$$\int f(x)\,dx.$$

We have seen that if $F(x)$ is a function whose derivative is $f(x)$, then

$$\int f(x)\,dx = \mathbf{F}(x) + C,$$

when C is a constant.

The process of finding an integral is called INTEGRATION; while the function $f(x)$ is found from $F(x)$ by *differentiation*, the function $F(x)$ is found from $f(x)$ by *integration*. Integration and differentiation are therefore "inverse" processes.

It may be noted here that a given function can always be differentiated, but it cannot always be integrated. To find indefinite integrals, we can, in fact, only work backward from known derivatives. For example:

since $\frac{d(x^3)}{dx} = 3x^2$, $\therefore \int x^2\,dx = \frac{1}{3}x^3 + C$;

and since $\frac{d(x^n)}{dx} = nx^{n-1}$, $\therefore \int x^{n-1}\,dx = \frac{1}{n}x^n + C$.

10. Differentiation of $\sin x$, $\cos x$, and $\tan x$.

(i) $y = \sin x$.

Let x be increased by δx and let δy be the increase in y, then

$$y + \delta y = \sin(x + \delta x),$$

and $\therefore$

$$\delta y = \sin(x + \delta x) - \sin x$$
$$= 2\cos(x + \tfrac{1}{2}\delta x)\sin\tfrac{1}{2}\delta x. \quad \text{(xix, p. 275)}$$

$$\therefore \frac{\delta y}{\delta x} = \cos(x + \tfrac{1}{2}\delta x)\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}.$$

Now when $\delta x \to 0$, $\cos(x + \frac{1}{2}\delta x) \to \cos x$ and $\frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x} \to 1$, *provided x is the number of radians in the angle.*

Since $\frac{\delta y}{\delta x}$ becomes $\frac{dy}{dx}$, we have therefore

$$\frac{dy}{dx} = \cos x,$$

$$\text{i.e.} \quad \boldsymbol{\frac{d(\sin x)}{dx} = \cos x.}$$

Since $\cos x = \sin(\pi/2 + x)$, this result may be stated in the form

$$\frac{d(\sin x)}{dx} = \sin(\pi/2 + x),$$

which means that the gradient of the graph of $\sin x$ at any value of x is given by the value of the ordinate at the point representing 90° farther along the axis of x.

Similarly, $\frac{d\sin(\pi/2 + x)}{dx} = \sin\{\pi/2 + (\pi/2 + x)\}$.

It follows at once, since $\sin(\pi/2 + x) = \cos x$, that if:

(ii) $y = \cos x$,

then

$$\frac{dy}{dx} = \frac{d}{dx}\sin(\pi/2 + x) = \sin(\pi + x) = -\sin x,$$

$$\text{i.e.} \quad \boldsymbol{\frac{d\cos x}{dx} = -\sin x.}$$

It follows that,

since $\frac{d(\sin x)}{dx} = \cos x$, $\therefore \int \cos x\, dx = \sin x + C$,

and since $\frac{d(\cos x)}{dx} = -\sin x$, $\therefore \int \sin x\, dx = -\cos x + C$.

EXAMPLES.

(1) At $x = 30°$, $\frac{d \sin x}{dx} = \cos x = \cos 30° = \frac{\sqrt{3}}{2} = \cdot 866$.*

(2) $\int_0^{30°} \cos x\, dx = \Big[\sin x + C\Big]_0^{30°} = \sin 30° - \sin 0 = \frac{1}{2}$.

(3) At $x = 30°$, $\frac{d \cos x}{dx} = -\sin x = -\sin 30° = -\frac{1}{2}$.

(4) $\int_0^{30°} \sin x\, dx = \Big[-\cos x + C\Big]_0^{30°} = -\cos 30° - (-\cos 0)$

$= -\cdot 866 + 1 = \cdot 134$.

(iii) $y = \tan x$.

Proceeding as before,

$$\delta y = \tan(x + \delta x) - \tan x$$

$$= \frac{\tan x + \tan \delta x}{1 - \tan \delta x \tan x} - \tan x$$

$$= \frac{\tan x + \tan \delta x - \tan x + \tan \delta x \tan^2 x}{1 - \tan \delta x \tan x}$$

(since δx is small we can write δx for $\tan \delta x$)

$$= \frac{\delta x + \delta x \tan^2 x}{1 - \delta x \tan x},$$

$$\therefore \frac{\delta y}{\delta x} = \frac{1 + \tan^2 x}{1 - \delta x \tan x}.$$

In the limit when δx becomes 0, $\delta x \tan x = 0$.

$$\therefore \frac{dy}{dx} = 1 + \tan^2 x = \sec^2 x.$$

The corresponding integral is $\int \sec^2 x\, dx = \tan x + C$.

* Since, in all these formulæ of differentiation and integration, x is measured in radians, we ought strictly to write $x = \pi/6$ instead of $x = 30°$.

11. In some cases differential coefficients can be deduced by Geometry.

EXAMPLE: $y = \sqrt{a^2 - x^2}$.

Since $x^2 + y^2 = a^2$, the graph is the upper half of the circumference of a circle, the centre of which is the origin and the radius a (fig. 5).

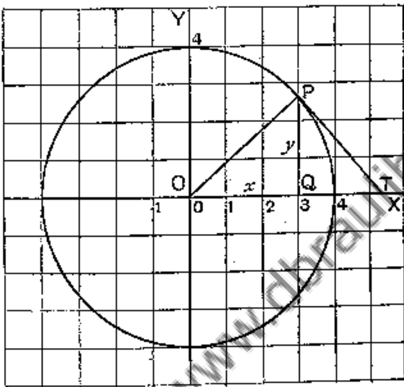

Fig. 5

Take any point P, co-ordinates x, y, on the circumference, and through it draw a tangent PT, cutting the axis of x at T.

Then the gradient of the graph at P is $\frac{QP}{TQ}$.

At P in the figure, the gradient is negative.
Now triangle PQT is similar to triangle OQP; therefore

$$\frac{QP}{TQ} = \frac{QO}{QP} = \frac{-x}{y} = \frac{-x}{\sqrt{a^2 - x^2}}.$$

Hence, if $y = \sqrt{a^2 - x^2}$,

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}} \text{ and } \int \frac{-x}{\sqrt{a^2 - x^2}} dx = \sqrt{a^2 - x^2} + C.$$

EXERCISE XXVII (c)

1. Find the differential coefficients of $\sin x$, $\cos x$, and $\tan x$ when x is 0°, 30°, 45°, 60°, 90° and 120°.

2. Draw graphs illustrating that (i) $\int a\,dx = ax + C$; (ii) $\int x^2dx = \frac{1}{3}x^3 + C$.

3. Find the area bounded by the graph of x^3 from $x = 0$ to $x = 10$, and from $x = 2$ to $x = 10$.

4. Find:

$$\text{(i)} \int_0^{45°} \sin x\,dx; \quad \text{(ii)} \int_0^{45°} \cos x\,dx; \quad \text{(iii)} \int_{30°}^{60°} \sin x\,dx;$$

$$\text{(iv)} \int_{30°}^{60°} \cos x\,dx; \quad \text{(v)} \int_{30°}^{45°} \sec^2 x\,dx.$$

5. Draw the graph of $y = \sin x$ from $x = 0$ to $x = 2\pi$ radians, and directly below, using the same scale, the graph of $\frac{d}{dx}\sin x$, i.e. $\cos x$, and below that the graph of $\frac{d}{dx}\cos x$. For what values of x is the gradient 0 in each graph?

6. Find the area bounded by the sine graph from (i) $x = 0$ to $x = 45°$; (ii) from $x = 30°$ to $60°$; (iii) from $x = 0$ to $x = 90°$.

7. Find $\frac{dy}{dx}$ when $y = \sqrt{25 - x^2}$ when x has the following values: (i) ± 3; (ii) ± 4; (iii) ± 5; (iv) 0.

8. Find the value of $\int_3^4 \frac{x\,dx}{\sqrt{(a^2 - x^2)}}$ when $a = 5$.

9. Show that $\int \tan^2 x\,dx = \tan x - x$.

10. Show that $\frac{d\cos x}{dx} = \cos(\frac{1}{2}\pi + x)$.

12. Sum and Difference, Product and Quotient of Functions.

(1) *Sum and difference.*

Let $y = u + v - w$, where u, v, w are functions of x. To find $\frac{dy}{dx}$.

Let u, v, w and y increase by δu, δv, δw, δy when x increases by δx.

$$\therefore \delta y = \delta u + \delta v - \delta w,$$

and
$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} - \frac{\delta w}{\delta x},$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}. \quad \ldots \ldots \quad \text{(I)}$$

(2) Similarly, $\frac{d}{dx}(ay) = a\frac{dy}{dx}$, if a is constant. . . . (II)

(3) *Product.*

Let $y = uv$, u and v being functions of x (e.g. $y = x^2 \sin x$).

To find $\frac{dy}{dx}$.

Let u, v and y increase to $(u + \delta u)$, $(v + \delta v)$ and $(y + \delta y)$ respectively when x is increased by δx.

Then $$(y + \delta y) = (u + \delta u)(v + \delta v)$$
$$= uv + v\,\delta u + (u + \delta u)\delta v,$$

Subtract $y = uv$.

$$\therefore \quad \delta y = v\,\delta u + (u + \delta u)\delta v$$

and $$\frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + (u + \delta u)\frac{\delta v}{\delta x}.$$

When $\delta x \to 0$, δu also $\to 0$, and $u + \delta u$ becomes u; $\frac{\delta y}{\delta x}$, $\frac{\delta u}{\delta x}$ and $\frac{\delta v}{\delta x}$ become respectively $\frac{dy}{dx}$, $\frac{du}{dx}$ and $\frac{dv}{dx}$.

Then $$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}. \quad . \; . \; . \; . \; . \quad \text{(III)}$$

Example i.—$y = x^2 \sin x$.

$$\frac{dy}{dx} = \sin x\frac{d(x^2)}{dx} + x^2\frac{d(\sin x)}{dx}$$
$$= 2x \sin x + x^2 \cos x.$$

Example ii.—$y = \sin 2x$
$$= 2 \sin x \cos x.$$
$$\frac{dy}{dx} = 2\left(\cos\frac{d \sin x}{dx} + \sin x\frac{d \cos x}{dx}\right)$$
$$= 2(\cos^2 x - \sin^2 x)$$
$$= 2 \cos 2x.$$

The corresponding integral is $\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C$.

(4) *Quotient.*

Let
$$y = \frac{u}{v},$$
$$y + \delta y = \frac{u + \delta u}{v + \delta v};$$
$$\therefore \quad \delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$
$$= \frac{v\,\delta u - u\,\delta v}{v(v + \delta v)}.$$

Dividing by δx,
$$\frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v + \delta v)}.$$

Proceeding to the limit, since $(v + \delta v)$ becomes v,
$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \quad \ldots\ldots \quad \text{(IV)}$$

Example i.
$$y = \frac{x^2}{\sin x},$$
$$\frac{dy}{dx} = \frac{2x\sin x - x^2\cos x}{\sin^2 x}.$$

Example ii.
$$y = \frac{\sin x}{\cos x}, \text{ i.e. } \tan x,$$
$$\frac{dy}{dx} = \frac{\cos x\dfrac{d\sin x}{dx} - \sin x\dfrac{d\cos x}{dx}}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The corresponding integral is,
$$\int \sec^2 x\,dx = \tan x + C,$$
and since $\sec^2 x = 1 + \tan^2 x$,
$$\int \tan^2 x\,dx = \tan x - \int dx$$
$$= \tan x - x.$$

EXAMPLE iii.

$$y = \frac{1}{x},$$

$$\frac{dy}{dx} = \frac{0 - 1}{x^2} = -\frac{1}{x^2} = -x^{-2}.$$

(5) Other useful forms of results (III) and (IV) are obtained by dividing (III) by $y = uv$, and (IV) by $y = \frac{u}{v}$.

From (III), if $y = uv$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx}. \quad \dots \dots \text{(V)}$$

From (IV), if $y = \frac{u}{v}$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} - \frac{1}{v}\frac{dv}{dx}. \quad \dots \dots \text{(VI)}$$

The derivative of any continued product can be found by (V).
Thus, if $y = u \,.\, v \,.\, w = (uv)w$,

then $$\frac{1}{y}\frac{dy}{dx} = \frac{1}{uv}\frac{d(uv)}{dx} + \frac{1}{w}\frac{dw}{dx},$$

or $$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}.$$

EXAMPLE i.

$$y = x^n = x \,.\, x \,.\, x \,.\, \dots\dots \text{ (n factors)}$$

Since $$\frac{dx}{dx} = 1, \ \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x} + \frac{1}{x} + \text{etc.} \dots \text{ (n terms)}$$

$$= \frac{n}{x}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{n}{x}y = \frac{nx^n}{x} = nx^{n-1}.$$

EXAMPLE ii.

$$y = \frac{1}{x^n}.$$

By (VI), $$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1} \times 0 - \frac{1}{x^n}nx^{n-1} = -\frac{n}{x}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{n}{x}y = -\frac{n}{x} \times \frac{1}{x^n} = -\frac{n}{x^{n+1}}.$$

Exercise XXVII (d)

Differentiate the following, using as many forms as you can.

1. $\dfrac{\sin x}{x}$. 2. $\dfrac{x}{\sin x}$. 3. $\cot x$. 4. $\dfrac{x^2}{\cos x}$. 5. $\dfrac{\cos x}{x^2}$.
6. $\dfrac{k}{x^n}$. 7. kx^n. 8. $\sin^2 x$. 9. $\cos^2 x$. 10. $\dfrac{1}{\sqrt[3]{x}}$.
11. $x^2 \sin x$. 12. $x \sin x \cos x$. 13. $\sin^n x$. 14. $\cos^n x$.

13. Function of a Function.

The function $\sin(x^2)$, for example, is the sine, not of x itself but of x^2, which is a function of x. The derivative of a function like $\sin(x^2)$ is easily found as follows.

Let $$y = f(z),$$
where $$z = F(x).$$

In the above example, $y = \sin z$, where $z = x^2$.

Let x become $x + \delta x$; then z becomes $z + \delta z$, since $z = F(x)$; and y becomes $y + \delta y$, since $y = f(z)$.

Now we always have, by simple algebra,

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \times \frac{\delta z}{\delta x}.$$

Take the limit of both sides when $\delta x \to 0$.

$$\therefore \quad \boldsymbol{\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}}.$$

Similarly, since $\dfrac{\delta x}{\delta y} \times \dfrac{\delta y}{\delta x} = 1$, we find

$$\frac{dx}{dy}\frac{dy}{dx} = 1,$$
or
$$\frac{dx}{dy} = 1\Big/\frac{dy}{dx}.$$

Example i.—$y = \sin(x^2)$.

Here $$y = \sin z, \text{ where } z = x^2.$$
$$\frac{dy}{dz} = \cos z; \quad \frac{dz}{dx} = 2x.$$
$$\therefore \quad \frac{dy}{dx} = \cos z \times 2x$$
$$= 2x \cos(x^2).$$

EXAMPLE ii.—$y = \sin 2x$.

Here $$y = \sin z, \text{ where } z = 2x.$$

$$\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx}$$
$$= \cos z \times 2$$
$$= 2\cos 2x.$$

Similarly, if $y = \sin mx$,

$$\frac{dy}{dx} = m\cos mx.$$

EXAMPLE iii.—$y = \sqrt{(x^2 - 3x + 2)}$.

Here $$y = \sqrt{z} = z^{\frac{1}{2}}, \text{ where}$$
$$z = x^2 - 3x + 2.$$

$$\frac{dy}{dx} = \frac{dy}{dz}\,\frac{dz}{dx}$$
$$= \tfrac{1}{2}z^{-\frac{1}{2}}(2x - 3)$$
$$= \tfrac{1}{2}\frac{2x - 3}{\sqrt{(x^2 - 3x + 2)}}.$$

EXAMPLE iv.—Derivative of y^n, y being a function of x.

Let $z = y^n$.

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}$$
$$= ny^{n-1}\frac{dy}{dx},$$

which can be found when y itself is given.

EXAMPLE v.—If $y^2 = 4ax$, find $\frac{dy}{dx}$.

Take the derivative of both sides with respect to x. Then, by example (iv),

$$2y\frac{dy}{dx} = 4a,$$

$$\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2\sqrt{ax}} = \sqrt{\frac{a}{x}}.$$

EXAMPLE vi.—If $\dfrac{d^2x}{dt^2} = -k^2x$,

prove that

$$\left(\frac{dx}{dt}\right)^2 + k^2x^2$$

is independent of t.

Differentiate with respect to t, obtaining $2\dfrac{dx}{dt}\cdot\dfrac{d^2x}{dt^2} + 2k^2x\dfrac{dx}{dt}$, and substitute.

EXERCISE XXVII (E)

Using relation $\dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx}$ in Exercises 1 to 6, find $\dfrac{dy}{dx}$.

1. $y = (2x - 3)^3$, putting $z = 2x - 3$.
2. $y = (3x^2 + 2x - 3)^4$.
3. $y = \sqrt{(2x^2 - 1)}$, putting $z = 2x^2 - 1$.
4. $y = \sqrt{a^2 - x^2}$.
5. $y = \dfrac{1}{\sqrt{(a^2 - x^2)}}$.
6. $y = \sin^3 x$, $y = \sin\frac{1}{2}x$, $y = \cos\frac{1}{2}x$.
7. Find $\dfrac{dy}{dx}$ when,

(i) $y^2 = a^2 + x^2$. (iv) $y^2 = a^2 - x^2$.

(ii) $y^3 = 3ax^2$. (v) $y = \dfrac{x^{\frac{1}{3}}}{3} - \dfrac{2}{\sqrt{x}}$.

(iii) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. (vi) $y = \tan\frac{1}{2}x$.

14. Power Series.

We have seen (p. 344) that a *finite* sum, such as

$$y = a + bx + cx^2 + \ldots + kx^n$$

can be differentiated *term by term*, so that

$$\frac{dy}{dx} = b + 2cx + \ldots + nkx^{n-1}.$$

It does not follow that the derivative of every function given by a series having an *infinite* number of terms can be obtained in this way. There is one very important type of series, however, for which differentiation term by term is possible, viz. convergent power series.

The simplest example of a convergent power series is the geometrical progression

$$y = 1 + x + x^2 + x^3 + \ldots ad\,\infty.$$

We know (p. 310) that this series is convergent if x lies between -1 and $+1$, these values themselves being excluded, and in fact the sum is $\frac{1}{1-x}$. Now it is a fact—which we may state without proof—that in this case we have

$$\frac{dy}{dx} = 1 + 2x + 3x^2 + \ldots ad\,\infty.$$

EXAMPLE.—Find the sum of the series just written. We have

$$y = \frac{1}{1-x} = (1-x)^{-1}.$$

As in last section, put $1 - x = z$; $\therefore\ y = z^{-1}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz}\frac{dz}{dx}\\ &= (-1)z^{-2}(-1)\\ &= \frac{1}{(1-x)^2}.\end{aligned}$$

Therefore $\quad 1 + 2x + 3x^2 + \ldots = \frac{1}{(1-x)^2}.$

This can be verified by ordinary algebra. We might now differentiate again, and thus find a series for $(1-x)^{-3}$, and so on. Compare section **21**, p. 359, on the Binomial Theorem.

15. The Exponential Series.

Consider the power series

$$y = a + bx + cx^2 + dx^3 + ex^4 + \ldots (ad\,\infty).$$

This gives

$$\frac{dy}{dx} = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 + \ldots$$

By pairing the terms, it is seen that the series for dy/dx will equal the series for y, provided $b = a$, $2c = b$, $3d = c$, $4e = d$, and so on.

Let $a = 1$, then $b = 1$,

$$\text{and } c = \tfrac{1}{2},$$

$$d = \frac{c}{3} = \frac{1}{1\,.\,2\,.\,3} = \frac{1}{3!},$$

$$e = \frac{d}{4} = \frac{1}{1\,.\,2\,.\,3\,.\,4} = \frac{1}{4!}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Thus if $$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots, \quad . \quad . \quad . \quad \text{(I)}$$

then, differentiating,

$$\frac{dy}{dx} = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots, \quad . \quad . \quad . \quad \text{(II)}$$

$$\text{i.e. } \frac{dy}{dx} = y.$$

The series found is a function whose rate of change is equal to the function itself. It is known as the **exponential function.**

The nth term is $\dfrac{x^{n-1}}{(n-1)!}$ and the next term is $\dfrac{x^n}{n!}$. The ratio of the latter to the former is $\dfrac{x^n}{n!}\Big/\dfrac{x^{n-1}}{(n-1)!} = \dfrac{x}{n}$, and for any value of x this can be made as small as we please by making n large enough—that is, by taking a large enough number of terms. Also the ratio becomes less as n increases. The series, beginning at $\dfrac{x^n}{n!}$, will then be less than

$$\frac{x^n}{n!}\left\{1 + \left(\frac{x}{n}\right) + \left(\frac{x}{n}\right)^2 + \left(\frac{x}{n}\right)^3 + \ldots\right\},$$

that is, less than $$\frac{x^n}{n!}\,\frac{1}{1 - x/n}.$$

The value of the series when x is 1 is always denoted by the letter e, and can be calculated to any desired place of decimals:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

$$\begin{aligned} &= 2{\cdot}5 \\ &\quad\ {\cdot}166666 \ldots \\ &\quad\ {\cdot}041666 \ldots \\ &\quad\ {\cdot}008333 \ldots \\ &\quad\ {\cdot}001388 \ldots \\ &\quad\ {\cdot}000198 \ldots \\ &\quad\ {\cdot}000024 \ldots \\ &\quad\ {\cdot}000002 \ldots \\ &\quad\ \overline{2{\cdot}718277} \\ &= 2{\cdot}71828 \quad \text{to five places of decimals.} \end{aligned}$$

16. Product of two exponential series.

If we take the series in x, and multiply it by the series in z, we get the series in $(x + z)$.

Carry out the multiplication with a few terms and arrange the resulting terms according to degree, thus:

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) \times \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)$$

$$= 1 + (x + z) + \left(\frac{x^2}{2!} + xz + \frac{z^2}{2!}\right) + \left(\frac{x^3}{3!} + \frac{x^2z}{2!} + \frac{xz^2}{2!} + \frac{z^3}{3!}\right) + \ldots$$

$$= 1 + (x + z) + \frac{(x + z)^2}{2!} + \frac{(x + z)^3}{3!} + \ldots$$

With the help of the Binomial Theorem, it can easily be shown that the law suggested by these early terms is general, so that

$$\left.\begin{aligned} &\left(1 + x + \frac{x^2}{2!} + \ldots\right) \times \left(1 + z + \frac{z^2}{2!} + \ldots\right) \\ &\quad = 1 + (x + z) + \frac{(x + z)^2}{2!} + \ldots + \frac{(x + z)^n}{n!} + \ldots \end{aligned}\right\} \quad . \quad \text{(III)}$$

If, then, we denote the function $1 + x + \frac{x^2}{2!} + \ldots$ by exp. (x), we have

$$\text{exp. } (x) \,.\, \text{exp. } (z) = \text{exp. } (x + z). \quad . \quad . \quad . \quad \text{(IV)}$$

This is exactly like the Index Law,

$$a^x \times a^z = a^{x+z};$$

and we can prove in fact that exp. $(x) = e^x$, where e is the number 2·71828 . . ., just found, and x is any integer or fraction, positive or negative.

Hence $$\boldsymbol{e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots .}$$

The series for e^x is true for all values of x. For example, on changing x into $-x$,

$$e^{-x} = 1 - x + \frac{x^2}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots .$$

Also, as we have already seen (section **15**),

$$\boldsymbol{\frac{d}{dx}(e^x) = e^x.}$$

We can now express the xth power of any positive number a as a power series in x. We first express a as a power of e, viz.

$$a = e^{\log_e a};$$

this, in fact, is simply the definition of $\log a$ to base e.

Then $$a^x = (e^{\log_e a})^x$$
$$= e^{x \log_e a}.$$

In the exponential series, change x into $x \log_e a$. Thus

$$a^x = 1 + x \log_e a + \frac{(x \log_e a)^2}{1 \cdot 2} + \ldots .$$

The series for the xth power of a number is therefore simplest when the number is e. This is the basis of the great importance of e in higher mathematics.

Logarithms to base e are called *natural* logarithms, also *Napierian* logarithms, after John Napier of Merchiston (1550–1617), to distinguish them from common logarithms to base 10.

Example i.—Find $\log_e 10$ from tables of common logs.

$$\text{Let } \log_e 10 = x.$$
$$\therefore e^x = 10.$$

Take common logs. $\therefore\ x \log_{10} e = 1,$

$$\therefore\ x = \frac{1}{\log_{10} e} = \frac{1}{\log_{10} 2{\cdot}7183}$$

$$= \frac{1}{{\cdot}4343} = 2{\cdot}3026.$$

EXAMPLE ii.—Find $\log_{10} \mathrm{N}$ from logs to base e.

We have $$\mathrm{N} = e^{\log_e \mathrm{N}}.$$

Take logs to base 10.

$$\therefore\ \log_{10} \mathrm{N} = \log_e \mathrm{N} \times \log_{10} e.$$

But, as in Ex. (i), $\log_{10} e = \dfrac{1}{\log_e 10}$.

$$\therefore\ \log_{10} \mathrm{N} = \frac{\log_e \mathrm{N}}{\log_e 10}.$$

We shall prove below (p. 362) that logs to base e can be found from a power series.

17. $\dfrac{d}{dx} \log_e x = \dfrac{1}{x}$.

If $$y = \log_e x,$$

then $$x = e^y.$$

$$\therefore\ \frac{dx}{dy} = e^y.$$

Thus (by article 13)

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x},$$

as stated.

Thus also $$\int \frac{dx}{x} = \log_e x + \mathrm{C}.$$

This supplies the missing case in the result,

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + \mathrm{C},$$

which fails when $n = -1$.

18. Derivative of an Inverse Function.

Since, when $x = e^y$, we have $y = \log_e x$, the exponential function and the logarithmic function to base e are called *inverse* functions to each other. We see by the last example that we can always find the derivative of an inverse function when we know the derivative of the direct function itself. Very important examples are the inverse functions $\sin^{-1}x$ and $\tan^{-1}x$, i.e. the angles (in radians) whose sine is x, or whose tan is x.

Let
$$y = \sin^{-1}x.$$
$$\therefore\ x = \sin y,$$
$$\frac{dx}{dy} = \cos y \quad \text{(p. 341)}$$
$$= \sqrt{1-x^2}.$$
$$\therefore\ \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$
or
$$\boldsymbol{\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}}.$$

Again, let $y = \tan^{-1}x$.
$$\therefore\ x = \tan y.$$
$$\therefore\ \frac{dx}{dy} = \sec^2 y \quad \text{(p. 342)}.$$
$$\therefore\ \frac{dy}{dx} = \frac{1}{\sec^2 y}$$
$$= \frac{1}{1+x^2},$$
or
$$\boldsymbol{\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}}.$$

There are many angles whose sine has a given value x. To prevent ambiguity, it is always understood that $\sin^{-1}x$ means the angle (in radians) between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, whose sine is x. Similarly, $\tan^{-1}x$ is always taken to lie between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. $\text{Cos}^{-1}x$ sometimes occurs; it lies between 0 and π.

These formulæ for the derivatives of $\sin^{-1}x$ and $\tan^{-1}x$ are chiefly useful in integration, where they are of great importance. Thus:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C.$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}x + C.$$

19. Hyperbolic Functions.

The function $\frac{e^x + e^{-x}}{2}$ is named $\cosh x$, and is positive for both positive and negative values of x since $e^{-x} = \frac{1}{e^x}$.

The function $\frac{e^x - e^{-x}}{2}$ is named $\sinh x$ (pronounced shine x), and may be either positive or negative.

Verify that,

$$\text{(i)}\quad \cosh x + \sinh x = e^x.$$
$$\text{(ii)}\quad \cosh x - \sinh x = e^{-x}.$$
$$\text{(iii)}\quad \cosh^2 x - \sinh^2 x = 1.$$

The quotient $\frac{\sinh x}{\cosh x}$ is called $\tanh x$ (pronounced tank x) and equals $\frac{e^x - e^{-x}}{e^x + e^{-x}}$.

These functions are known as the **hyperbolic functions.**

Verify that $$\frac{d}{dx}\cosh x = \sinh x$$

and $$\frac{d}{dx}\sinh x = \cosh x,$$

and state the corresponding integrals.

20. Extensions.

(i) If $y = e^{kx}$, k being a constant, then

$$\frac{dy}{dx} = ke^{kx} = ky.$$

This is easily confirmed by writing e^{kx} in the series form $1 + kx + \frac{(kx)^2}{2!}$ + etc., and differentiating with respect to x.

An application of this important relation is the case in which the rate of change is proportional at every instant to the value of the thing changing, as, for example, in some forms of growth and leakage. If y is the value of the thing and x the time of the change, then $\frac{dy}{dx} = ky$. The above relation states that this condition is satisfied if $y = e^{kx}$.

The integral is $\int e^{kx} = \frac{1}{k} e^{kx} + C$.

(ii) Any number (N) can be expressed as a power of e, simply from the definition of a logarithm; thus:

$$N = e^{\log_e N}.$$

If $y = N^x = e^{(\log N)x}$, then $\frac{dy}{dx} = \log N e^{(\log N)x} = (\log N)y$.

The integral is $\int N^x dx = \frac{N^x}{\log_e N} + C$.

(iii) *Logarithmic Differentiation.*

If $y = \frac{u_1 u_2 \ldots}{v_1 v_2 \ldots}$, all functions of x,

then $\log y = \log u_1 + \log u_2 + \text{etc.} - \log v_1 - \log v_2 - \text{etc.}$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{1}{u_1}\frac{du_1}{dx} + \frac{1}{u_2}\frac{du_2}{dx} + \text{etc.} - \frac{1}{v_1}\frac{dv_1}{dx} - \frac{1}{v_2}\frac{dv_2}{dx} - \text{etc.},$$

which is in agreement with previous considerations (p. 347).

Example 1.—If $y = \sqrt{\frac{2x-3}{2x+3}}$, find $\frac{dy}{dx}$.

$$\log y = \tfrac{1}{2}\{\log(2x-3) - \log(2x+3)\},$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left\{\frac{2}{2x-3} - \frac{2}{2x+3}\right\} = \frac{6}{(2x-3)(2x+3)}.$$

$$\therefore \frac{dy}{dx} = \frac{6}{(2x-3)(2x+3)} \times y = \frac{6}{(2x-3)^{\frac{1}{2}}(2x+3)^{\frac{3}{2}}}.$$

EXAMPLE 2.—If $y = e^{-\frac{1}{2}x^2}\sin 2x$,

$$\log y = -\tfrac{1}{2}x^2 + \log\sin 2x.$$

$$\frac{1}{y}\frac{dy}{dx} = -x + \frac{2\cos 2x}{\sin 2x}.$$

$$\therefore \frac{dy}{dx} = -xe^{-\frac{1}{2}x^2}\sin 2x + 2e^{-\frac{1}{2}x^2}\cos 2x.$$

(iv) *Integration by Substitution.*

Beginning with $\int \frac{dy}{y} = \log y$:

1. Let $y = \sin x$, then $dy = \cos x\,dx$

and $$\therefore \int \frac{\cos x\,dx}{\sin x} = \log\sin x;$$

$$\text{i.e. } \int \cot x\,dx \quad\text{or}\quad \int\frac{dx}{\tan x} = \log\sin x.$$

2. Similarly, if $y = \cos x$,

$$\int \frac{\sin x\,dx}{\cos x} = -\log\cos x = \log\sec x,$$

$$\text{i.e. } \int \tan x\,dx = \log\sec x.$$

3. Since $$\frac{1}{\sin x} = \frac{1}{2\sin\frac{1}{2}x\cos\frac{1}{2}x} = \frac{\frac{1}{2}\sec^2\frac{1}{2}x}{\tan\frac{1}{2}x},$$

and $$\frac{d(\tan\frac{1}{2}x)}{dx} = \tfrac{1}{2}\sec^2\tfrac{1}{2}x,$$

$$\therefore \int\frac{dx}{\sin x} = \int\frac{d(\tan\frac{1}{2}x)}{\tan\frac{1}{2}x} = \log\tan\tfrac{1}{2}x.$$

4. Since $$\cos x = \sin\left(\frac{\pi}{2} + x\right),$$

$$\int\frac{dx}{\cos x} = \int\frac{dx}{\sin(\pi/2 + x)} = \log\tan\tfrac{1}{2}\left(\frac{\pi}{2} + x\right).$$

21. The Binomial Theorem.

The expansion of $(1 + x)^n$ can be established in a manner similar to that used for the exponential series. If n is a positive integer, the first term is obviously 1 and the last x^n.

Let $(1 + x)^n = 1 + ax + bx^2 + cx^3 + \ldots + x^n;$

then by differentiation

$$n(1 + x)^{n-1} = a + 2bx + 3cx^2 + \ldots.$$

To be true for all values of x it must be true for $x = 0$.

Putting
$$x = 0$$
$$a = n,$$

continuing the differentiation,

$$n(n - 1)(1 + x)^{n-2} = 2b + 2 . 3cx + 3 . 4dx^2 + \ldots.$$

For $x = 0$,

$$n(n - 1) = 2b \text{ from which } b = \frac{n(n - 1)}{2}.$$

Similarly

$$c = \frac{n(n - 1)(n - 2)}{3!}, \quad d = \frac{n(n - 1)(n - 2)(n - 3)}{4!}, \ldots.$$

$$\therefore (1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!}x^2 + \frac{n(n - 1)(n - 2)}{3!}x^3 + \ldots + x^n.$$

If $-x$ is substituted for x, then

$$(1 - x)^n = 1 - nx + \frac{n(n - 1)}{2!}x^2 - \frac{n(n - 1)(n - 2)}{3!}x^3, + \ldots \pm x^n,$$

the sign of x^n depending on whether n is even or odd.

For the expansion of $(x + y)^n$, we have:

$$(x + y)^n = x^n\left(1 + \frac{y}{x}\right)^n$$

$$= x^n\left(1 + n\frac{y}{x} + \frac{n(n - 1)}{2!}\frac{y^2}{x^2} + \ldots + \frac{y^n}{x^n}\right),$$

$$\boldsymbol{(x + y)^n = x^n + nx^{n-1}y + \frac{n(n - 1)}{2!}x^{n-2}y^2 + \ldots + y^n.}$$

Since
$$(y + x)^n = (x + y)^n,$$

it follows that the coefficients of terms equidistant from the ends of the expansion are the same.

E.g., the coefficients of $x^{(n-1)}y$ and $xy^{(n-1)}$ are both n.

22. More Expansions in Power Series.

1. Cos x and sin x.

The method just used for $(1 + x)^n$ can be applied to $\cos x$ and $\sin x$.

Let $\cos x = a + bx + cx^2 + dx^3 + ex^4 + \ldots$.

By successive differentiation

$$\begin{aligned} -\sin x &= b + 2cx + 3dx^2 + 4ex^3 + \ldots, \\ -\cos x &= 2c + 2.3dx + 3.4ex^2 + \ldots, \\ +\sin x &= 2.3d + 2.3.4ex + \ldots, \\ +\cos x &= + 2.3.4e + \ldots . \end{aligned}$$

Put $x = 0$ in these equations.

$$\therefore 1 = a;\ 0 = b;\ -1 = 2c;\ 0 = d;\ +1 = 2.3.4e;\ \ldots .$$

Thus, $$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad \text{(I)}$$

In the same way, or by differentiation of (I),

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots \quad \text{(II)}$$

The relationship of these expansions to that of e^x (p. 354) will be noticed. That relationship is very important, both in pure and in applied mathematics—in Electrical Engineering, for example. Like the series for e^x, the series for $\cos x$ and $\sin x$ are convergent for all values of x.

If we write the symbol i for $\sqrt{-1}$, we see that

$$e^{ix} = 1 + ix + \frac{i^2x^2}{1.2} + \frac{i^3x^3}{1.2.3} + \ldots$$

or $$e^{ix} = \cos x + i \sin x. \quad \text{(III)}$$

Similarly, $$e^{-ix} = \cos x - i \sin x.$$

$$\left.\begin{aligned} \therefore \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \end{aligned}\right\} \quad \text{(IV)}$$

If we tried to apply the above method to $\tan x$, we should soon find ourselves in difficulties. But we can find the series for $\tan x$ by ordinary algebraic division, since

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots}$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots .$$

2. Logarithms (see p. 355).

(*Note.*—Except in purely arithmetical work, logarithms in mathematics are always supposed taken to base e. If there is any possibility of doubt, we can of course write $\log_e x$.) We need not try to find a power series for $\log x$, since $\log 0$ is not a finite number. In fact, if N is a large positive number,

$$\log \frac{1}{N} = -\log N,$$

or, as a limit,

$$\log 0 = -\infty.$$

With $\log(1+x)$ there is no difficulty. We can here use a new method, which is often useful.

Put

$$y = \log(1+x);$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x}$$

$$= 1 - x + x^2 - x^3 + \ldots,$$

provided x is numerically less than 1, for convergence.

Then, by integration,

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots .$$

$$\therefore \log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots, \quad . \quad \text{(V)}$$

if $x^2 < 1$. No constant is needed, for both sides are 0 when $x = 0$.

Change x to $-x$. Then, if $x^2 < 1$,

$$\therefore \log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \ldots . \quad \text{(VI)}$$

Suppose we wish to find $\log_e 2$. We might, as an extreme case, put $x = 1$ in (V). The equation

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

is correct, but very unsuitable for calculation. A better method is: put $x = \frac{1}{2}$ in (VI). Then, since $\log \frac{1}{2} = -\log 2$,

$$\therefore \quad \log_e 2 = \tfrac{1}{2} + \tfrac{1}{2}\frac{1}{2^2} + \tfrac{1}{3}\frac{1}{2^3} + \tfrac{1}{4}\frac{1}{2^4} + \dots \quad . \quad \text{(VII)}$$

(Prove from this that $\log_e 2 = \cdot 6931 \dots$)

Next, we can get $\log_e 3$, by putting $x = \frac{1}{3}$ in (VI). Thus,

$$\log_e 3 - \log_e 2 = \tfrac{1}{3} + \tfrac{1}{2}\frac{1}{3^2} + \tfrac{1}{3}\frac{1}{3^3} + \tfrac{1}{4}\frac{1}{3^4} + \dots \quad \text{(VIII)}$$

$$\therefore \log_e 3 = 1{\cdot}0986.$$

We can now find the very important number $\log_e 10$, by putting $x = \frac{1}{10}$ in (VI). For this gives

$$\log_e 10 - 2\log_e 3 = \tfrac{1}{10} + \tfrac{1}{2}\frac{1}{10^2} + \tfrac{1}{3}\frac{1}{10^3} + \dots, \quad . \text{(IX)}$$

which leads to $\log_e 10 = 2{\cdot}3026.$

3. $\tan^{-1} x$.

We have
$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$
$$= 1 - x^2 + x^4 - x^6 + \dots,$$

(if $x^2 < 1$). Integrate.

$$\therefore \tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \dots,$$

($x^2 < 1$). No constant is needed.

Put $x = 1$.

$$\therefore \frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots .$$

Like the series for $\log_e 2$, viz.

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots,$$

this formula for $\frac{\pi}{4}$, though interesting, is not suitable for rapid calculation.

EXERCISE XXVII (F)

1. Compare the terms of the series

$$e - 1 = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

with those of the G.P.

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots,$$

and show that e is therefore less than 3.

2. Show that $$\frac{d(e^{-x})}{dx} = -e^{-x}$$

and that $\therefore \int e^{-x}dx = -e^{-x} + C.$

3. Find x such that $e^x = 6$, and express 6 as an exponential series.

4. If $y = (e^x \pm e^{-x})$, show that

$$\frac{dy}{dx} = (e^x \mp e^{-x}).$$

5. Show that $(e^x + e^{-x})^2 = e^{2x} + e^{-2x} + 2$, and that

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1.$$

6. If $y = \sqrt[3]{\dfrac{3x}{2x+3}}$, find $\dfrac{dy}{dx}$.

(Begin with $\log y = \frac{1}{3}\{\log 3x - \log(2x+3)\}$).

7. Verify the following:

(i) $\dfrac{d}{dx}\log\cos x = -\tan x.$

(ii) $\dfrac{d}{dx}\log\tan x = \dfrac{2}{\sin 2x}.$

(iii) $\dfrac{d}{dx}\log\dfrac{x}{1+x} = \dfrac{1}{x(1+x)}.$

(iv) $\dfrac{d}{dx}\log\dfrac{x}{\sqrt{(1+x^2)}} = \dfrac{1}{x(1+x^2)}.$

8. Differentiate:

(i) $\log(k+x)$; (ii) $\log(k-x)$; (iii) e^{-3x}; (iv) 3^x.

9. Find $\int e^{-3x}\,dx$, $\int 3^x\,dx$, and $\int_0^1 \frac{dx}{\sqrt{(1-x^2)}}$.

10. By putting $x = 1$ in the expansion of $(1+x)^n$, find the sum of the coefficients of the expansion.

11. If population (P) follows the law,

$$P_2 = P_1 e^{kt},$$

find k when $P_1 = 26$ millions, $P_2 = 29$ millions, and $t = 10$ years. Use this value of k to find the population in another 10 years.

12. If £1 increasing in value at every moment amounts to £1, 1s. in a year, find the rate of increase (i.e. k in the relation $1{\cdot}05 = e^{kx}$, where $x = 1$).

13. Find the amount of £1 increasing continually for 20 years when $k = {\cdot}05$, and compare it with the amounts at compound and simple interest at 5% per annum for the same time.

14. If water leaks from a tank according to the law $q = Qe^{-kt}$, in which q is the quantity remaining after time t, and if q is 1500 gallons when t is 0, and is 1200 gallons after 5 min., find the quantity remaining after 20 min., and in what time there is only a gallon left.

15. Show that

$$e^{x+\delta} = e^x\left(1 + \delta + \frac{\delta^2}{2!} + \text{etc.}\right),$$

and find the change in e^x when x increases by ·01 from the value found in Exercise 3.

16. Find $\frac{dy}{dx}$ when (i) $y = e^{\sin x}$, (ii) $y = x\tan^{-1}x$.

17. Show that:

(i) $(\cos x \pm i\sin x)^2 = \cos 2x \pm i\sin 2x$.

(ii) $(\cos x \pm i\sin x)^3 = \cos 3x \pm i\sin 3x$.

(iii) $(\cos x \pm i\sin x)^n = e^{\pm inx} = \cos nx \pm i\sin nx$.

(Demoivre, 1667–1754.)

CHAPTER XXVIII

APPLICATIONS OF THE CALCULUS, AND EXERCISES

1. Maximum and Minimum Values.

In your practical work on the functions, $ax^2 + bx + c$, $\sin x$, $\cos x$, etc., you have found that at the points of maximum and minimum value, the gradient is 0; that is, the differential coefficient is 0.

This fact enables us to determine the value of x for which the function is a maximum or a minimum.

EXAMPLE i.

$$y = ax^2 + bx + c,$$

$$\frac{dy}{dx} = 2ax + b.$$

For a minimum value, $\frac{dy}{dx} = 0.$

Hence $2ax + b = 0,$

from which $x = -\frac{b}{2a}$

and

$$y = \frac{b^2}{4a} - \frac{b^2}{2a} + c = \frac{-b^2}{4a} + c$$

$$= \frac{4ac - b^2}{4a}.$$

This agrees with the result on p. 221.

EXAMPLE ii.

$$y = \sin x,$$

$$\frac{dy}{dx} = \cos x.$$

For a maximum or minimum, $\cos x = 0$, which is true when x is

$$\frac{\pi}{2} \text{ or } \frac{3\pi}{2}.$$

Hence $\sin x$ is a maximum or a minimum when x is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Whether the value is a maximum or a minimum can be decided as follows. Refer to a graph of the function, e.g. $y = \sin x$ (fig. 1).

If regarded from left to right, i.e. in the positive direction of x, the gradient changes from + to −, and the value of y at the point for which $\frac{dy}{dx} = 0$ is a maximum; if the gradient changes from − to +, the value of y is a minimum. If the values of $\frac{dy}{dx}$ for the

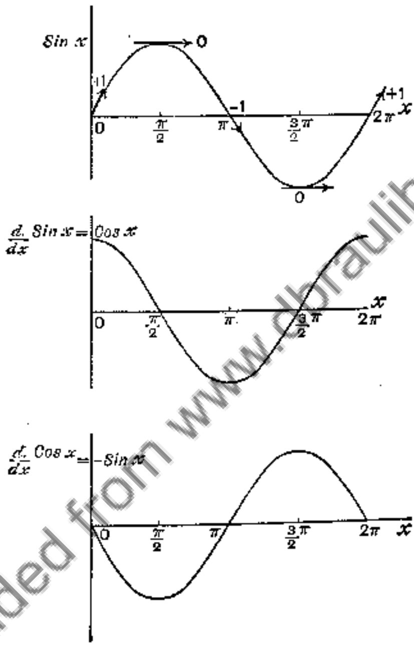

Fig. 1

function are graphed, this graph also will have a gradient, called the **second derivative** or **second differential coefficient** of the function.

If this second derivative is negative, the value of y is a maximum; if positive, a minimum.

This is borne out by the graphs of fig. 1 at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. The second derivative is written $\frac{d^2y}{dx^2}$.

In Example i, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2ax + b) = 2a$ which is +, if a is +.

$$\therefore \text{ at } x = -\frac{b}{2a}, y \text{ is a minimum.}$$

In Example ii, $\frac{d^2y}{dx^2} = -\sin x$, which at $\frac{\pi}{2}$ is negative (-1).

$$\therefore \text{ at } x = \frac{\pi}{2}, y \text{ is a maximum,}$$

whereas at $\frac{3\pi}{2}$, $-\sin x$ is positive $(-(-1) = +1)$.

$$\therefore \text{ at } x = \frac{3\pi}{2}, y \text{ is a minimum.}$$

It may be noted here that if $y = f(x)$, the second derivative $\frac{d^2y}{dx^2}$ is also written $f''(x)$, the first derivative $\frac{dy}{dx}$ being $f'(x)$.

Similarly, we have higher derivatives, $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots$, written $f'''(x), f^{(iv)}(x), \ldots$, when $y = f(x)$.

2. The Trajectory of a Projectile.

The equation given on p. 267 for the path of a projectile can be more easily established as follows:

$$y = ax^2 + bx; \quad \ldots \quad \text{(i)}$$

therefore

$$\frac{dy}{dx} = 2ax + b. \quad \ldots \quad \text{(ii)}$$

At the point $x = 0$, $\frac{dy}{dx}$ is the elevation of the gun, and substituting 0 for x, we have

$$\frac{dy}{dx} = b. \quad \ldots \quad \text{(iii)}$$

That is, the value of the tangent of the angle the gun makes with the horizontal is b,

$$\text{i.e.} \quad \tan e = b. \quad \ldots \quad \text{(iv)}$$

Again, the maximum altitude is $\frac{-b^2}{4a}$, and in terms of the velocity is $\frac{V^2 \sin^2 e}{2g}$.

Hence $$-\frac{b^2}{4a} = \frac{V^2 \sin^2 e}{2g},$$

from which $$a = -\frac{b^2 g}{2V^2 \sin^2 e}$$

$$= -\frac{g}{2V^2}\frac{\tan^2 e}{\sin^2 e}, \text{ from (iv)},$$

$$= -\frac{g}{2V^2 \cos^2 e}$$

$$= -\frac{g}{2V^2}\sec^2 e.$$

Equation (i) then becomes

$$y = -\frac{gx^2}{2V^2}\sec^2 e + x \tan e.$$

3. Small Differences.

The fact that $\frac{\delta y}{\delta x}$ is nearly equal to $\frac{dy}{dx}\left(\text{written } \frac{\delta y}{\delta x} \approx \frac{dy}{dx}\right)$ can be used to estimate small differences approximately.

Example i.—Find the change in x^4 when x changes from 3 to 3·01.

Here $\delta x = \cdot 01$.

Let $y = x^4$, and let δy be changed in y;

then $$\frac{\delta y}{\delta x} \approx \frac{dy}{dx} = 4x^3;$$

$$\therefore \quad \delta y \approx 4x^3 \delta x = 4 \times 27 \times \cdot 01 = 1{\cdot}08.$$

Example ii.—Find the change in $\cos x$ when x changes from 30° to 30° 20′.

$$\delta x = 20' = \cdot 0058 \text{ radians.}$$

$$\delta y \approx \frac{dy}{dx} \,.\, \delta x = -\sin x \,.\, \delta x$$

$$= -\cdot 5 \times \cdot 0058$$

$$= -\cdot 0029,$$

$$\text{i.e. } \cos 30^\circ\, 20' = \cdot 866 - \cdot 0029$$

$$= \cdot 8631.$$

This agrees with the tables.

Application to Equations.

EXAMPLE iii.—The following illustrates a method of finding approximate solutions of equations. (For the volumes required, see sections 7 and 8, Chap. XXII.)

A conical wine-glass, depth 8 cm. and diameter of rim 12 cm., contains liquid to a depth of 4 cm. A sphere of diameter 6 cm. is placed inside (fig. 2). Find by how much the liquid rises.

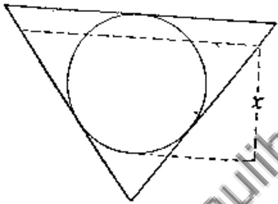

Fig. 2

By similar triangles it is found that the centre of the sphere is 5 cm. above the bottom of the wine-glass. The lowest point is therefore 2 cm. above the bottom of the glass. (See Ex. 12, p. 127.)

Let x (cm.) = the vertical distance between the lowest point of the sphere and the surface of the liquid.

The volume of liquid between the sphere and the wine-glass, reckoned from the horizontal plane of the lowest point of the sphere, is the same as the volume of the wine-glass from depth 2 to depth 4 cm., namely, $\frac{1}{3}\pi(3^2 \times 4 - 1\frac{1}{2}^2 \times 2) = 10\frac{1}{2}\pi$ (c.c.).

The volume of liquid between the sphere and the wine-glass sides is also

$$\tfrac{1}{3}\pi[\{\tfrac{3}{4}(2 + x)\}^2(2 + x) - (\tfrac{3}{2})^2 \times 2] - \tfrac{1}{3}\pi x^2(3^2 - x)$$

$$= \pi\frac{36x - 30x^2 + 8\frac{1}{3}x^3}{16}.$$

Equating,

$$\pi\frac{36x - 30x^2 + 8\frac{1}{3}x^3}{16} = 10\tfrac{1}{2}\pi,$$

which reduces to the cubic equation

$$\frac{36x - 30x^2 + 8\frac{1}{3}x^3}{16} = 10\tfrac{1}{2}.$$

It is evident that x lies between $(4 - 2)$ and $(8 - 2)$, i.e. between 2 and 6.

On trying 4, the left-hand side of the equation is found to equal $12\frac{1}{3}$. Its correct value is $10\frac{1}{2}$, which is $(12\frac{1}{3} - 1\frac{5}{6})$.

Let $\dfrac{36x - 30x^2 + 8\frac{1}{3}x^3}{16} = y$.

The problem now is to find what change from 4 in x will make a change of $-1\frac{5}{6}$ in y.

We have $\dfrac{dy}{dx} = \dfrac{36 - 60x + 25x^2}{16}$, which for $x = 4$ is $12\frac{1}{4}$.

From $\dfrac{dy}{dx} \approx \dfrac{\delta y}{\delta x}$,

$$\delta x = \delta y \Big/ \frac{dy}{dx} = \frac{-1\frac{5}{6}}{12\frac{1}{4}} = -0{\cdot}15 \text{ nearly.}$$

The new value of x is thus $4 - 0{\cdot}15 = 3{\cdot}85$. Substituting this value of x, y is found to be 10·7, which is nearly 10·5.

If desired, a still more exact value can be found by using 3·85 for x and $-0{\cdot}2$ for δy.

The first result gives 1·85 cm. for the rise of the surface.

Exercise XXVIII (a)

1. Find a near solution to the equation, $x^3 + 6x^2 + 5x - 11{\cdot}5 = 0$. Find the remaining roots by dividing by (x − found root) and solving the resulting quadratic equation.
2. Find a near root of the equation, $2x^3 - 3x^2 + x + 8 = 15$. Show that the other roots are imaginary.
3. Find by graph an approximate solution to the equation
$$x^4 + 4x^3 + 2x^2 - x + 5 = 0.$$
Then find a nearer value of x.
4. Find a closer solution of the equation, $x + \sin x = \dfrac{\pi}{4}$, given that for $x = 23°$, $x + \sin x = {\cdot}7921$.
5. Find $\cos 45° \, 20'$, without tables.
6. Turn to Exercise XX (a), No. 14. Find the angle of elevation or quadrant angle of the gun.
7. Find the maximum value of $1 + 6x - x^3$, for positive values of x, and state the corresponding value of x.
8. Arrange the equation $y = \dfrac{-gx^2}{2v^2}\sec^2 e + x \tan e$ in a convenient form for finding e; then find the angle of elevation

(e) of a gun in order that an object at an altitude (y) 1000 ft. and at a horizontal distance (x ft.) of 2000 yd. may be hit, the muzzle velocity (v) being 2500 ft. per second. Account for the two answers.

9. Turn back to Exercise XX (A), p. 222, and solve exercises, from 7 to 12, by differentiating the expressions.

10. Find the dimensions of the rectangle which has a maximum area for a given perimeter p.

11. Find the dimensions of a cylinder such that for a combined circumference and length of 9 ft., the volume may be a maximum.

4. Expansion of a Gas.

When a gas expands *at constant temperature*, the change is said to be **isothermal**, and the relation between pressure and volume is given by the equation

$$pv = \text{K}, \text{ where K is a constant.}$$

It follows that $p = \text{K}/v$.

The work done when the gas expands from a volume v_1 to a volume v_2 is the limit of the sum $\Sigma(p\,\delta v)$ between these limits —the pressure changing according to the law, $pv = \text{K}$ (see Chap. XXVII, section **9**).

Hence, $$\text{Work} = \int_{v_1}^{v_2} p\,dv = \int_{v_1}^{v_2} \frac{\text{K}}{v}\,dv = \text{K}\Big[\log_e v\Big]_{v_1}^{v_2}$$
$$= \text{K}(\log_e v_2 - \log_e v_1)$$
$$= \text{K}\log_e \frac{v_2}{v_1}.$$

K is found from any simultaneous values of p and v.

When *heat is not allowed to enter or to escape from* the gas, the change is said to be **adiabatic**, and the relation between pressure and volume is: $pv^s = \text{K}$, from which $p = \dfrac{\text{K}}{v^s}$.

The work done when the gas expands from v_1 to v_2 is:

$$\int_{v_1}^{v_2} p\,dv = \int_{v_1}^{v_2} \frac{\text{K}}{v^s}\,dv = \int_{v_1}^{v_2} \text{K}v^{-s}\,dv = \text{K}\left[\frac{v^{(1-s)}}{1-s}\right]_{v_1}^{v_2}$$
$$= \frac{\text{K}}{1-s}\left(v_2^{(1-s)} - v_1^{(1-s)}\right)$$
$$= \frac{\text{K}}{s-1}\left(\frac{1}{v_1^{(s-1)}} - \frac{1}{v_2^{(s-1)}}\right).$$

If the pressure of the gas is p_1 when the volume is v_1, and p_2 when the volume is v_2, this result reduces to $\frac{1}{s-1}(p_1v_1 - p_2v_2)$, since $p_1v_1^s = K$, and $p_2v_2^s = K$.

For most gases, s equals 1·41.

EXAMPLE.—Find the work done when a gas at a pressure of 100 lb. per sq. in., and of volume 1·5 c. ft., expands to a volume 3·5 c. ft., (i) at constant temperature, (ii) adiabatically.

5. The Circumference and Area of a Circle.

(1) Consider any sector.

Let the angle contained by the two radii be θ radians, the radius R and the arc y.

Then
$$y = R\theta. \qquad \text{(i)}$$

Now let the angle be increased by a small amount $\delta\theta$, and let the corresponding change in y be δy.

Then
$$y + \delta y = R(\theta + \delta\theta). \qquad \text{(ii)}$$

Subtracting equation (i) from equation (ii), we have

$$\delta y = R\,\delta\theta \text{ (the increment in the arc),}$$

and therefore
$$\frac{\delta y}{\delta\theta} = R.$$

When δy and $\delta\theta$ are diminished indefinitely, we obtain

$$\frac{dy}{d\theta} = R,$$

i.e. the rate of change of the arc with respect to the angle it subtends at the centre is equal to the radius (a constant).

It follows that $\int R\,d\theta = y$ (the arc), and this is true for all values of θ. Therefore,

$$\begin{aligned}\text{Circumference of circle} &= \int_0^{2\pi} R\,d\theta \\ &= R\Big[\theta\Big]_0^{2\pi} \\ &= 2\pi R.\end{aligned}$$

(2) Similarly, since the area of a small sector of angle $\delta\theta$ is $\frac{1}{2}R^2\delta\theta$,

$$\text{Area of circle} = \int_0^{2\pi} \tfrac{1}{2}R^2\,d\theta$$
$$= \tfrac{1}{2}R^2\Big[\theta\Big]_0^{2\pi}$$
$$= \pi R^2.$$

Show, as in (1), that the rate of change of area of a sector of a circle with respect to the angle is $\frac{1}{2}R^2$.

(3) In the following example an important integral is established. Let $2c$ denote the length of a chord at distance x from one end of the diameter at right angles to it.

The area between this chord and a parallel chord δx from it is $2c\,\delta x$, and the area of the segment from $x = 0$ is $2\int_0^x c\,dx$, which, since $c = \sqrt{x(2R - x)}$, is

$$2\int\sqrt{x(2R - x)}\,dx.$$

If 2θ is the angle at the centre subtended by the chord,

$$\sin\theta = \frac{c}{R} = \frac{\sqrt{x(2R - x)}}{R}.$$

Now the segment is the difference between the sector whose angle is 2θ and the triangle whose base is the chord and whose vertex is the centre of the circle.

$$\therefore \text{ Area of segment} = R^2\theta - (R - x)\sqrt{x(2R - x)}.$$

But
$$\theta = \sin^{-1}\frac{\sqrt{x(2R - x)}}{R}.$$

$$\therefore \int_0^x \sqrt{x(2R - x)}\,dx$$
$$= \frac{1}{2}\left\{R^2\sin^{-1}\frac{\sqrt{x(2R - x)}}{R} - (R - x)\sqrt{x(2R - x)}\right\}.$$

Verify that for $x = R$ this result equals $\frac{1}{4}\pi R^2$.

By taking x to be the distance of the chord from the centre, it can be shown in a similar manner that:

$$\int\sqrt{R^2 - x^2}\,dx = \frac{1}{2}\left\{R^2\sin^{-1}\frac{x}{R} + x\sqrt{R^2 - x^2}\right\}.$$

6. Average Value of the Sine Function Ordinates from 0 to $\frac{\pi}{2}$.

The problem is really, "What is the height of the rectangle on the base 0 to $\frac{\pi}{2}$, whose area is equal to that bounded by the sine graph from 0 to $\frac{\pi}{2}$?"

$$\text{Area bounded by sine graph} = \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= \Big[-\cos x\Big]_0^{\frac{\pi}{2}}$$

$$= \left[-\cos\frac{\pi}{2} - (-\cos 0)\right]$$

$$= [0 - (-1)]$$

$$= 1;$$

$$\therefore \text{ area of rectangle} = 1.$$

Since base of rectangle $= \frac{\pi}{2}$, height of rectangle $= \frac{1}{\pi/2} = \frac{2}{\pi}$,

$$\text{i.e. average value} = \frac{2}{\pi} = \cdot 636.$$

(i) Find the average value of $\cos x$ from $x = 0$ to $x = \frac{\pi}{2}$.

(ii) Find the average value of $\sin x$ from $x = 0$ to $x = \pi$.

(iii) Find the average value of $\cos x$ from $x = 0$ to $x = \pi$.

(iv) Of what angle is the sine equal to the average value of the sine from 0 to $\frac{\pi}{2}$?

7. Root Mean Square.

The average of the square of the ordinates of the sine curve is useful in Engineering.

(i) It is readily deduced as follows:

Since the ordinates of the sine curve have the same value from 0 to $\frac{\pi}{2}$ as those of the cosine curve from $\frac{\pi}{2}$ to 0, the squares of the corresponding ordinates must be the same, and the average of the squares of the ordinates of these two curves must be the same,

$$\text{e.g. } \sin 30^\circ = \cos(90^\circ - 30^\circ)$$

and $$\sin^2 30^\circ = \cos^2(90^\circ - 30^\circ)$$

(see fig. 3).

Now, $$\sin^2 x + \cos^2 x = 1,$$

and this is true for all values of $\sin^2 x$ and $\cos^2 x$, and true, therefore, for their average values, which we have seen are equal.

Hence,

$$2\sin^2 x = 1 \quad \text{and} \quad 2\cos^2 x = 1,$$

where x denotes the angle at which the squares of $\sin x$ and of $\cos x$ have their average value.

Therefore, $\sin^2 x = \frac{1}{2}$, and also $\cos^2 x = \frac{1}{2}$,

i.e. the average of the squares of the ordinates of the sine or cosine curve is $\frac{1}{2}$.

The root of the average (or mean) square, frequently denoted by **r.m.s.**, is $\sqrt{\frac{1}{2}} = 0{\cdot}707$.

For what angle have the sine and cosine this value?

(ii) The average value may be calculated as follows:

$$\text{Average value of } \sin^2 x = \frac{\int_0^{2\pi} \sin^2 x\,dx}{2\pi}. \quad \ldots \quad \text{(i)}$$

Now, $$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x;$$

therefore

$$\int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) dx = \tfrac{1}{2}\int_0^{2\pi} dx - \tfrac{1}{2}\int_0^{2\pi} \cos 2x\,dx$$

$$= \pi - \tfrac{1}{4}\Big[\sin 2x\Big]_0^{2\pi}$$

$$= \pi - 0$$

$$= \pi.$$

Hence, equation (i) becomes:

$$\text{Average value of } \sin^2 x = \frac{\pi}{2\pi}$$

$$= \tfrac{1}{2}.$$

EXERCISE.—Show that in (ii) the same result is obtained when the limits of the integration are $\frac{\pi}{2}$ and 0 and the divisor in equation (i), $\frac{\pi}{2}$.

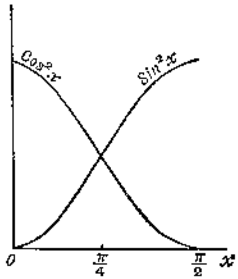

Fig. 3

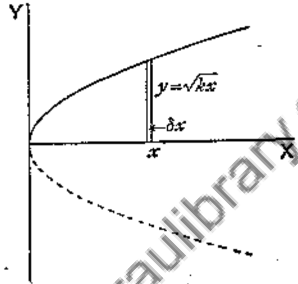

Fig. 4

8. Volume of a Paraboloid.

If the parabola $y^2 = kx$ (see p. 386) revolves about the axis of x, a solid called a paraboloid (of revolution) is generated.

The strip of thickness δx, at a distance x, generates a disc of volume $\pi y^2 \delta x$, or $\pi kx \delta x$, which shows how the volume changes with respect to x (fig. 4).

The volume of the whole paraboloid of length x is therefore

$$\int_0^x \pi kx\,dx = \tfrac{1}{2}\pi kx^2.$$

EXAMPLE i.—Show that $\frac{1}{2}\pi kx^2$ is half the product of the base and altitude of the paraboloid.

EXAMPLE ii.—A solid wooden cylinder is hollowed out so that the interior is a hollow paraboloid. What fraction of the solid cylinder remains?

9. Simpson's 1, 4, 2, 4, 1 Rule for Plane Areas.

The equation of the bounding curve is taken to be

$$y = a + bx + cx^2.$$

Consider three equidistant ordinates y_1, y_2, y_3, the middle one being at $x = 0$ and the others at $x = -s$ and $x = +s$ respectively (fig. 5); then

$$y_1 = a - bs + cs^2,$$
$$y_2 = a,$$
$$y_3 = a + bs + cs^2,$$

from which $$bs = \frac{y_3 - y_1}{2}$$

and $$cs^2 = \frac{y_1 + y_3 - 2a}{2} = \frac{y_1 + y_3 - 2y_2}{2}.$$

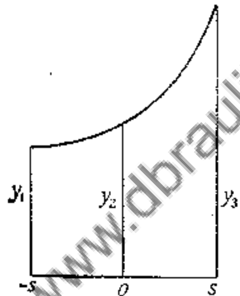

Fig. 5

Area bounded by the curve, extreme ordinates and axis of x

$$= \int_{-s}^{+s} y\,dx$$
$$= \int_{-s}^{+s} (a + bx + cx^2)\,dx = {}_{-s}^{\;\;s}\left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]$$
$$= \left(as + \frac{bs^2}{2} + \frac{cs^3}{3}\right) - \left(-as + \frac{bs^2}{2} - \frac{cs^3}{3}\right)$$
$$= s\left(2a + \frac{2cs^2}{3}\right)$$
$$= \frac{s}{3}(6a + 2cs^2)$$
$$= \frac{s}{3}(6y_2 + y_1 + y_3 - 2y_2)$$
$$= \frac{s}{3}(y_1 + 4y_2 + y_3).$$

If five ordinates are taken the result is,

$$\frac{s}{3}\{(y_1 + 4y_2 + y_3) + (y_3 + 4y_4 + y_5)\}$$

$$= \frac{s}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 + y_5), \text{ and so on.}$$

Hence the rule:

Divide the area into an even number of parts by an odd number of equidistant ordinates. Take the sum of the first and last ordinates, twice the sum of the remaining odd ordinates, and four times the sum of the even ordinates; add the three results together, multiply the total by the common distance between the ordinates and divide by three.

Notes.—The first or last ordinate or both may be 0.

A quick method is to mark successively the ordinates of each set on a strip of paper, measure the total length by ruler and multiply by the appropriate number (2 or 4).

EXAMPLE.—Plot the graph of $y = 2x + 3x^2$ and by Simpson's rule find the area under the graph from $x = 1$ to $x = 5$.

10. Volume of a Sphere.

The ordinate (y) of the semicircle, shown in fig. 6, at a distance x from the end of the diameter, is given by the equation

$$\begin{aligned} y^2 &= x(D - x) \\ &= Dx - x^2. \end{aligned}$$

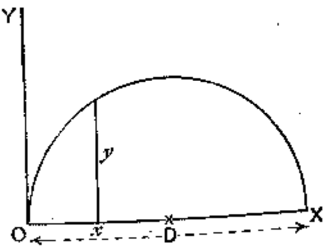

Fig. 6

If the semicircle revolves about the diameter it generates a sphere, and the ordinate y, a circle of area πy^2. Consider a very thin disc or zone of the sphere of thickness δx at this position. Its volume is $\pi y^2\,\delta x$ or $\pi(\mathrm{D}x - x^2)\,\delta x$.

$$\therefore \text{ Volume of the whole sphere} = \int_0^{\mathrm{D}} \pi(\mathrm{D}x - x^2)\,dx$$

$$= \pi\mathrm{D}\int_0^{\mathrm{D}} x\,dx - \pi\int_0^{\mathrm{D}} x^2\,dx$$

$$= \pi\mathrm{D}\left[\frac{x^2}{2}\right]_0^{\mathrm{D}} - \pi\left[\frac{x^3}{3}\right]_0^{\mathrm{D}}$$

$$= \frac{\pi\mathrm{D}^3}{2} - \frac{\pi\mathrm{D}^3}{3}$$

$$= \frac{\pi\mathrm{D}^3}{6} \text{ or } \tfrac{4}{3}\pi\mathrm{R}^3.$$

(Compare this with the statements on p. 246.)

Similarly, the volume of a cap of thickness t

$$= \int_0^t \pi(\mathrm{D}x - x^2)\,dx$$

$$= \pi\mathrm{D}\frac{t^2}{2} - \pi\frac{t^3}{3}$$

$$= \pi t^2(\mathrm{R} - \tfrac{1}{3}t),$$

as at end of section 8, Chap. XXII.

The volume of any solid of revolution may be found in the same way. If V is the volume from a fixed section up to the section at x, then $\dfrac{d\mathrm{V}}{dx} = \pi y^2$.

11. Surface of a Sphere.

Consider a thin sector at an angle θ from the diameter of the semicircle (fig. 7).

The small arc $\mathrm{R}\,\delta\theta$ is subtended by a small angle $\delta\theta$ at the centre of the circle. This arc sweeps out a narrow belt of the surface of the sphere generated when the semicircle revolves.

Then the radius of the belt is $\mathrm{R}\sin\theta$ and the circumference $2\pi\mathrm{R}\sin\theta$.

The area of the narrow belt is $2\pi\mathrm{R}\sin\theta \times \mathrm{R}\,\delta\theta = 2\pi\mathrm{R}^2\sin\theta\,\delta\theta$.

$$\text{Area of the surface of the sphere} = \int_0^{\pi} 2\pi\mathrm{R}^2\sin\theta\,d\theta$$

(The limits 0 to π include the whole semicircle, and therefore the whole sphere)

$$= 2\pi R^2 \Big[-\cos\theta\Big]_0^\pi \dagger$$
$$= -2\pi R^2(\cos\pi - \cos 0)$$
$$= -2\pi R^2(-1 - 1)$$
$$= 4\pi R^2.$$

If the integration is made between the limits 0 and θ such that $\cos\theta = \frac{R - t}{R}$, we obtain the curved surface of a spherical cap of thickness t (see the figure).

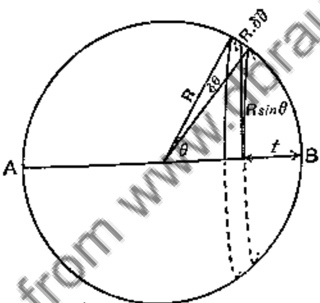

Fig. 7

From the line marked †, we have:

Curved surface of spherical cap $= 2\pi R^2 \Big[-\cos\theta\Big]_0^\theta$
$$= -2\pi R^2(\cos\theta - \cos 0)$$
$$= -2\pi R^2\left(\frac{R - t}{R} - 1\right)$$
$$= -2\pi R^2\left(-\frac{t}{R}\right)$$
$$= 2\pi Rt.$$

Exercise XXVIII (b)

1. Find quickly a near answer to the increase in volume and in surface when a soap bubble increases in diameter from 10 to 10·1 cm.
2. Find the area of the cap of a sphere of 10 cm. radius, the curve of the dome of which measures 12 cm.
3. Find the area of the ocean surface within a circle of radius 1000 miles measured along the arc of a great circle.
4. A spherical buoy, diameter 4 ft., sinks from 2 ft. 6 in. to 2 ft. 7 in. Find the change in displacement.
5. A conical buoy, diameter 6 ft. and height 8 ft., sinks from 5 ft. to 5 ft. 2 in. Find the change in displacement.

12. Volume of any Solid.

We have proved (p. 338) that if A is the area between a curve, the axis of x, a fixed ordinate at $x = a$, and the ordinate at x, then

$$\frac{dA}{dx} = y$$

and

$$A = \int_a^x y\,dx.$$

The volume of any solid can be found by a similar method. Let $A(x)$ be the area of the cross-section by a plane perpendicular to Ox, at the section x; and let $V(x)$ be the volume between a fixed section at $x = a$, and the section at x. Then $\delta V(x) = A(x)\,\delta x$, approximately, when δx is small.

Thus

$$\frac{dV(x)}{dx} = A(x),$$

or

$$\frac{dV}{dx} = A;$$

and

$$V = \int_a^x A\,dx.$$

Volume of a pyramid.

Suppose, for example, the base of the pyramid to be a triangle ABC, O being the vertex (fig 8).

Let abc be the section at distance x from O. If the perpendicular to the base from O meets the section abc at d, and the base at D, then $\triangle abc$ is similar to $\triangle$ABC, and therefore

$$A(x) = \text{area } abc = \text{area ABC} \times \frac{x^2}{p^2} = \Delta \frac{x^2}{p^2},$$

where p = OD.

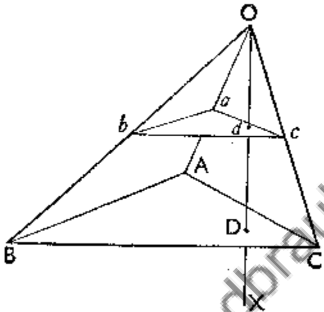

Fig. 8

If volume Oabc = V(x), then

$$\frac{dV(x)}{dx} = A(x)$$

$$= \frac{\Delta}{p^2} x^2.$$

$$\therefore V(x) = \frac{\Delta}{p^2} \frac{x^3}{3},$$

no constant being required, since V(0) = 0.

$\therefore$ Volume of whole pyramid (for which $x = p$)

$$= \tfrac{1}{3}\Delta p$$
$$= \tfrac{1}{3} \text{ area of base} \times \text{altitude}.$$

Exercise XXVIII (c)

1. A cylindrical drum is 5 ft. long, and the diameter of its end is 4 ft. When the drum is on its side, it contains oil to a depth of $1\frac{1}{2}$ ft. Find what volume run off will lower the surface by half an inch.

2. If the rate at which water flows into a conical vessel is proportional to the depth x, i.e. $\frac{dV}{dt} = cx$, find an expression for the time t in terms of x.

3. Find expressions as in Ex. 2, when

$$\text{(i) } \frac{dV}{dt} = c\sqrt{x}; \quad \text{(ii) } \frac{dV}{dt} = cx^2.$$

4. A funnel is emptying at the rate of $4\sqrt{x}$ c.c. per sec., x being the depth of the liquid at the time t. Find the time taken, the dimensions of the funnel being: diam. of rim 20 cm., depth 15 cm.

5. If $\frac{dx}{dt} = k - x$, and $x = 0$ when $t = 0$, prove that

$$t = \log_e \frac{k}{k - x},$$
$$x = k(1 - e^{-t}).$$

CHAPTER XXIX

CONIC SECTIONS

1. Definitions.—Conic sections, or conics, were originally defined by the Greeks as the curves of section of a right circular cone by planes in various positions. They are more conveniently defined as loci of a point moving in a plane, thus:

A conic is the locus of a point which moves so that the ratio of its distance from a fixed point to its distance from a fixed straight line is constant.

The fixed point is the *focus*, the fixed straight line the *directrix*, and the constant ratio the *eccentricity*, e.

The three kinds of conics are shown in fig. 1, in which e is respectively $\frac{3}{5}$, 1, $\frac{5}{3}$,

$$SP_1 = \tfrac{3}{5} P_1M_1;\ SP_2 = P_2M_2;\ SP_3 = \tfrac{5}{3} P_3M_3.$$

When $e < 1$, the conic is called an *ellipse*;
when $e = 1$, a *parabola*;
when $e > 1$, a *hyperbola*.

Note that the ellipse is a closed oval; the parabola is a single curve whose ends extend to infinity; and the hyperbola has two branches extending to infinity (see fig. 6).

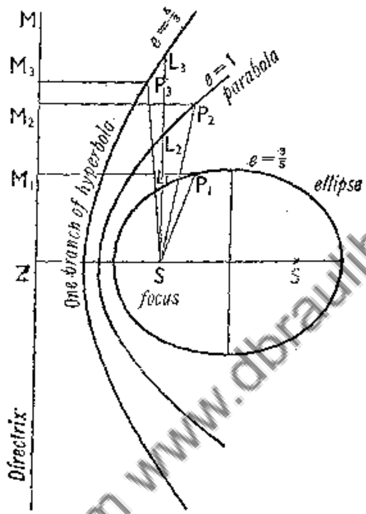

Fig. 1

2. The Parabola.

The object is to choose suitable axes and to express the relation between the co-ordinates (x, y) of any point P on the conic by an equation.

In fig. 2 the directrix is ZM, the focus S, the vertex A, and P is any point on the conic.

Since $\frac{AS}{AZ} = e$ and $e = 1$, AS = AZ.

Choose AS as the axis of x, and the perpendicular through the vertex A as the axis of y. The co-ordinates of any point P are x = AN and y = NP.

Let AZ and therefore AS = a. Then ZN = $(a + x)$ and NS = $(a - x)$.

Now, since $\frac{SP}{PM} = e$ and $e = 1$, SP = PM, and since PM = ZN, SP = $(a + x)$.

By the Pythagoras theorem $NP^2 = SP^2 - NS^2$,

$$\text{i.e.} \quad y^2 = (a + x)^2 - (a - x)^2,$$

which reduces to $$y^2 = 4ax.$$

This is the required equation to the PARABOLA. It is seen that since $y = \pm 2\sqrt{ax}$ there are two values of y for each positive value of x.

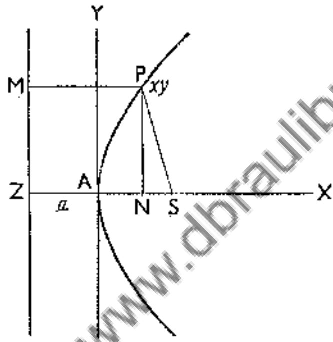

Fig. 2

The double ordinate at the focus S is called the **latus-rectum** of the conic. In this case the value of x for S is a, and $y = \pm\sqrt{4a^2} = \pm 2a$. Hence the latus-rectum of the parabola is $4a$.

Important Features of the Parabola $y^2 = 4ax$.

1. The focus is at $x = +a$, directrix at $x = -a$.
2. The semi-latus-rectum is $2a$.
3. The focal distance of any point $P(x\,y)$ on the parabola is $(a + x)$.

Mechanical Construction of the Parabola.

The parabola may be drawn mechanically by fixing one end of a length of thin string or thread to a place (X) near the edge of the blade of a Tee-square, and the other end at a focus (S) on a sheet of paper pinned to the drawing-board (fig. 3). The string is kept taut and in contact with the edge of the blade by the pencil, while the Tee-square is slid down the edge of the board. The pencil marks out a parabola.

The directrix is at a distance from A equal to AS, and since XM = xZ is the length of the string, it is obvious that PM = SP in every position of the pencil.

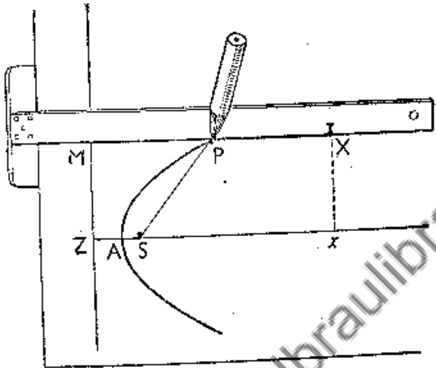

Fig. 3

3. The Ellipse.

In this case the eccentricity e is less than unity.

It appears from figs. 1 and 4, and might easily be proved, that the figure is symmetrical about line CY perpendicular to SZ.

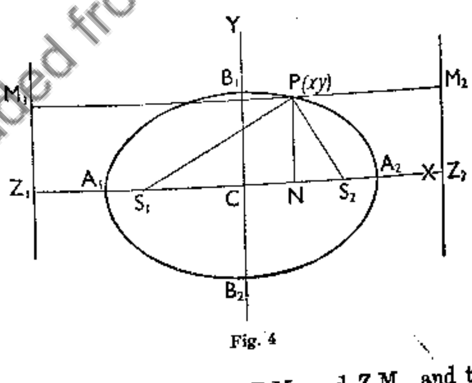

Fig. 4

It has therefore two directrices Z_1M_1 and Z_2M_2, and two foci S_1 and S_2, so situated (fig. 4) with respect to the ellipse that

$$A_1Z_1 = A_2Z_2 \text{ and } A_1S_1 = A_2S_2.$$

For point A_1, we have

$$A_1S_1 = eA_1Z_1. \quad \ldots \ldots \quad \text{(I)}$$

But another point A_2 on the other side of S_1 also satisfies this condition.

(For example, if $e = \frac{3}{5}$, then (i) $A_1S_1 = 3$, and $A_1Z_1 = 5$, and (ii) $A_2S_1 = 12$ and $A_2Z_1 = 20$ satisfy this value of e.)

$$\therefore\ A_2S_1 = eA_2Z_1. \quad \ldots \ldots \quad \text{(II)}$$

By addition of I and II, since $A_2Z_2 = A_1Z_1$,

$$A_1A_2 = eZ_1Z_2, \quad \ldots \ldots \quad \text{(III)}$$

and by subtraction of I from II, since $A_1S_1 = A_2S_2$,

$$S_1S_2 = eA_1A_2. \quad \ldots \ldots \quad \text{(IV)}$$

Take A_1A_2 as the axis of x, and through C, the middle point of A_1A_2, draw the axis of y.

Let $A_1A_2 = 2a$.

From equation III, $2a = eZ_1Z_2 = 2eCZ_2$.

$$\therefore\ CZ_2 = \frac{a}{e}. \quad \ldots \ldots \quad \text{(V)}$$

From equation IV, $2CS_2 = 2ea$.

$$\therefore\ CS_2 = ea. \quad \ldots \ldots \quad \text{(VI)}$$

If $P(xy)$ is any point on the ellipse, $CN = x$, $NP = y$,

$$S_2P = ePM_2 = eNZ_2 = e(CZ_2 - CN) = e\left(\frac{a}{e} - x\right) = (a - ex).$$

Similarly, $$S_1P = e\,.\,NZ = a + ex.$$

Hence $$\mathbf{S_1P + S_2P = 2a}, \quad \ldots \ldots \quad \text{(VII)}$$

a very important property.

Since PNS_2 is a right-angled triangle,

$$NS_2^2 + NP^2 = S_2P^2,$$

i.e. $$(ea - x)^2 + y^2 = (a - ex)^2,$$

$$e^2a^2 + x^2 - 2eax + y^2 = a^2 + e^2x^2 - 2eax,$$

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2).$$

$$\therefore\quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \quad \ldots \ldots \quad \text{(VIII)}$$

This is one form of the equation to the ellipse.

$$\text{At } x = 0, y = \pm a\sqrt{1 - e^2}.$$

That is, the other semi-axes CB_1 and CB_2 are $+a\sqrt{1 - e^2}$ and $-a\sqrt{1 - e^2}$ respectively. If we put $a\sqrt{1 - e^2} = b$, the equation becomes,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (IX)$$

This equation expresses the relation between x and y in terms of the semi-axes, a and b.

Important Features of the Ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

1. The semi-axes are $\pm a$ and $\pm b$, $b = a\sqrt{1 - e^2}$. The greater axis is called the major axis, and the less, the minor axis.

2. The foci are at $x = +ea = +\sqrt{a^2 - b^2}$, and $x = -ea = -\sqrt{a^2 - b^2}$.

3. The semi-latus-rectum is the value of y at the foci (i.e. at $x = \pm\sqrt{a^2 - b^2}$), namely, $\pm \frac{b^2}{a}$.

4. The directrices are at $x = \pm \frac{a}{e} = \pm \frac{a^2}{\sqrt{(a^2 - b^2)}}$.

5. The focal distances S_1P and S_2P of any point $P(xy)$ on the ellipse are $(a + ex)$ and $(a - ex)$ respectively.

6. The sum of the focal distances of any point P is $2a$ (the length of the major axis) and is therefore constant.

7. If $b = a$, the equation becomes that of the circle $x^2 + y^2 = a^2$. This circle, whose centre is C and radius a, is called the *auxiliary circle.*

Mechanical Construction of the Ellipse.

In this construction use is made of the property (VII, p. 388) that the sum of the focal distances of any point is constant and equal to $2a$, the major axis.

A pin or tack is fixed at each focus (fig. 5) and a loop of thread placed over them. The thread is kept taut as a pencil point P describes the curve about the foci.

If the thread keeps constant in length, the sum of the focal distances S_1P and S_2P remains unaltered, since S_1S_2 does not change.

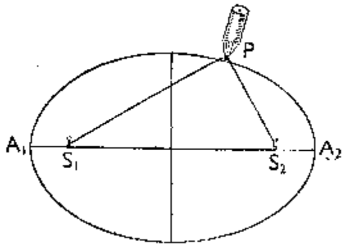

Fig. 5

EXAMPLE.—If $2l$ is the length of the thread, show that $l = a(1 + e)$.

4. The Hyperbola.

The eccentricity e is greater than unity.

In fig. 6, Z_1M_1 is a directrix and S_1 a focus.

The vertex A_1 of the hyperbola is situated such that

$$S_1A_1 = eA_1Z_1. \qquad \text{(I)}$$

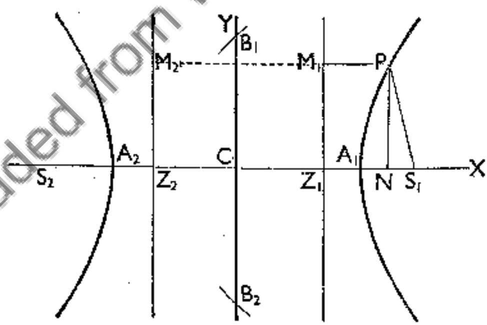

Fig. 6

Here again, a second point A_2, but this time on the same side of S_1, in S_1Z_1 produced, satisfies the condition

$$S_1A_2 = eA_2Z_1. \qquad \text{(II)}$$

Take the straight line CY bisecting A_1A_2 at right angles as the axis of y, and CA_1S_1 as the axis of x.

In A_1A_2 produced take point S_2 such that

$$A_2S_2 = A_1S_1,$$

and draw a directrix Z_2M_2 through Z_2, such that

$$A_2Z_2 = A_1Z_1.$$

Let $A_1A_2 = 2a$.
By adding equations I and II,

$$S_1S_2 = eA_1A_2, \text{ since } A_2S_2 = A_1S_1.$$
$$\therefore\ 2CS_1 = 2ea,$$

from which $$CS_1 = ea;$$

and by subtracting I from II

$$A_1A_2 = eZ_1Z_2, \text{ since } A_2Z_2 = A_1Z_1.$$
$$\therefore\ 2a = 2eCZ_1,$$

from which $$CZ_1 = \frac{a}{e}.$$

If $P(xy)$ is any point on the curve, $CN = x$ and $NP = y$,

$$S_1P = ePM_1 = eNZ_1 = e(CN - CZ_1) = e\left(x - \frac{a}{e}\right) = ex - a.$$

$$NS_1 = (CS_1 - CN) = (ea - x).$$

Since PNS_1 is a right-angled triangle,

$$S_1P^2 = NP^2 + NS_1^2,$$
$$\text{i.e. } (ex - a)^2 = y^2 + (ea - x)^2.$$
$$\therefore\ x^2(e^2 - 1) - y^2 = a^2(e^2 - 1).$$

Dividing through by $a^2(e^2 - 1)$,

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$$

If b^2 is written for $a^2(e^2 - 1)$, the equation becomes

$$\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

Comparing the equations of the ellipse and the hyperbola it is seen that they differ in one sign only, and many relations of the hyperbola can be obtained from those of the ellipse by substituting $-b^2$ for $+b^2$.

It will be seen that fig. 6 consists of two branches, one on the right side of the axis of y, and the other on the left.

The equation proves this, since for any value of y there is a positive and also a negative value of x.

The two branches of the hyperbola and their foci and directrices are interchangeable.

Important Features of the Hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

1. For $x = 0$, $y = \pm\sqrt{-b^2}$. This root cannot be extracted. The hyperbola does not cut the axis of y, but if the axis is cut at B_1 and B_2 by a circular arc described with centre A_1 and radius ea, then $CB_1 = +a\sqrt{e^2 - 1} = b$ and $CB_2 = -a\sqrt{e^2 - 1} = -b$. B_1B_2 is called the conjugate axis of the hyperbola, the other axis being A_1A_2.

2. The foci are at $x = ea = +\sqrt{a^2 + b^2}$ and $x = -ea = -\sqrt{a^2 + b^2}$.

3. The semi-latus-rectum is the value of y for $x = \sqrt{a^2 + b^2}$, namely, $\frac{b^2}{a}$.

4. The directrices are at $x = \pm\frac{a}{e} = \pm\frac{a^2}{\sqrt{(a^2 + b^2)}}$.

5. The focal distances S_1P and S_2P of any point $P(xy)$ on the hyperbola are $(ex - a)$ and $(ex + a)$ respectively.

6. The difference between the focal distances of any point P is $2a$, and is therefore constant.

7. If $b = a$, $e = \sqrt{2}$, and the equation becomes $x^2 - y^2 = a^2$. The hyperbola in this case is called an equilateral or rectangular hyperbola.

Mechanical Construction of the Hyperbola.

The device is based on the property that the difference between the focal distances of any point P is constant. It consists (fig. 7) of a rod S_2R pivoted at one focus S_2. Near the other end of the rod is a hole through which a string is passed and knotted. The other end of the string is made fast to a pin or tack at the other focus S_1. The pencil used to trace the curve is placed so as to keep more and more of the string close up against the rod as the pencil moves along the rod from R towards S_2, and the rod swings across. This is equivalent to subtracting equal lengths from the rod and the string. The difference between the effective part of the rod and

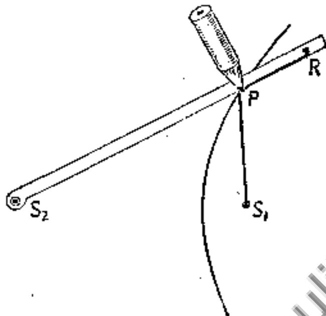

Fig. 7

of the string is unaltered by subtracting the same length (PR) from each.

Exercise XXIX (A)

1. Determine the chief features of the conics,

$$y^2 = 12x,$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1,$$
$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

2. Find the eccentricity e of the conics,

$$\frac{x^2}{25} + \frac{y^2}{4} = 1 \quad \text{and} \quad \frac{x^2}{25} - \frac{y^2}{4} = 1.$$

3. Find the points of intersection of

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{x^2}{16} - \frac{y^2}{9} = 1,$$

and interpret the result.

4. Find the points of intersection of

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \quad \text{and} \quad y^2 = 6x.$$

5. The axes of an ellipse are respectively 1 in. and $1\frac{1}{2}$ in. Find the eccentricity, and the positions of the foci and directrices.

5. Tangents to the Conics.

(i) The *parabola*, $y^2 = 4ax$.

The gradient at any point is $\frac{dy}{dx}$.

By differentiation, $$2y\frac{dy}{dx} = 4a.$$

$$\therefore \frac{dy}{dx} = \frac{2a}{y}.$$

At the point x_1y_1 the gradient is $\frac{2a}{y_1}$.

The equation to the tangent at x_1y_1 is the straight-line equation,

$$y - y_1 = \frac{2a}{y_1}(x - x_1).$$

$$\therefore yy_1 - y_1^2 = 2a(x - x_1),$$

which, since $y_1^2 = 4ax_1$, reduces to

$$\boldsymbol{yy_1 = 2a(x + x_1)}.$$

(ii) The *ellipse*, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Proceeding as in (i), first find $\frac{dy}{dx}$.

Since $$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

$$2y\frac{dy}{dx} = -2\frac{b^2}{a^2}x.$$

$$\therefore \frac{dy}{dx} = -\frac{b^2}{a^2}\cdot\frac{x}{y}.$$

At the point x_1y_1 the gradient is $-\frac{b^2}{a^2}\frac{x_1}{y_1}$, and the equation to the tangent therefore

$$y - y_1 = -\frac{b^2}{a^2}\frac{x_1}{y_1}(x - x_1),$$

$$yy_1 - y_1^2 = -\frac{b^2}{a^2}(xx_1 - x_1^2).$$

Transposing, $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

And since $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1,$$

the equation to the tangent at x_1y_1 is

$$\boldsymbol{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.}$$

(iii) The *hyperbola*, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Proceeding exactly as in (ii), the equation is found to be

$$\boldsymbol{\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.}$$

It will be observed that the equations to the tangents may be written down from the equations to the conics by substituting xx_1 for x^2, yy_1 for y^2, and in the case of the parabola $(x + x_1)$ for $2x$.

Example.—Find the equation to the tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the points for which $x = 3$.

From the equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$ the values of y for $x = 3$ are found to be $\pm 2{\cdot}4$.

The equation to the tangent to the ellipse is

$$\frac{xx_1}{25} + \frac{yy_1}{9} = 1,$$

which for $x_1 = 3$ and $y_1 = 2{\cdot}4$ becomes

$$\frac{3x}{25} + \frac{4y}{15} = 1, \quad \text{or} \quad y = 3{\cdot}75 - {\cdot}45x,$$

and for $x_1 = 3$ and $y_1 = -2{\cdot}4$,

$$\frac{3x}{25} - \frac{4y}{15} = 1, \quad \text{or} \quad y = {\cdot}45x - 3{\cdot}75.$$

6. Asymptotes of the Hyperbola.

These are two straight lines which the hyperbola approaches but never meets (fig. 8).

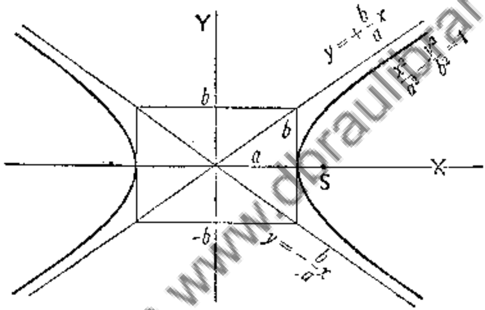

Fig. 8

From the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$= \frac{x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right).$$

$$\therefore y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}.$$

If x^2 is very great, $\sqrt{\left(1 - \frac{a^2}{x^2}\right)}$ is nearly equal to 1, so that the value of y, at a point on the curve for which x is large, is nearly $\pm\frac{b}{a}x$.

The value of $\sqrt{1 - \dfrac{a^2}{x^2}}$ is nearly $1 - \dfrac{1}{2}\dfrac{a^2}{x^2}$, as we may see by squaring; therefore, approximately,

$$y = \pm \frac{b}{a} x \left(1 - \frac{a^2}{2x^2}\right)$$

$$= \pm \frac{b}{a} x \mp \frac{ab}{2x^2}.$$

The last term becomes very small as $x^2 \to \infty$. Hence the y co-ordinate of a point on the curve is nearly equal to $\pm \frac{b}{a} x$,

i.e. $$y \approx \pm \frac{b}{a} x.$$

The two lines $y = \pm \frac{b}{a} x$ are the asymptotes.

In the case of the rectangular hyperbola, $b = a$, and the equations to the asymptotes become $y = +x$ and $y = -x$.

These straight lines make 45° with the axis of x, since $\tan 45° = 1$ and $\tan(-45°) = -1$, and are therefore at right angles to each other. This accounts for the name "rectangular hyperbola".

It may easily be shown from a figure that the lengths of the perpendiculars from a point (x, y) to the lines $x - y = 0$ and $x + y = 0$ are $\dfrac{x - y}{\sqrt{2}}$ and $\dfrac{x + y}{\sqrt{2}}$ respectively. The product of these perpendiculars is $\frac{1}{2}(x^2 - y^2)$, which is equal to $\frac{1}{2}a^2$, if the point (x, y) is on the rectangular hyperbola.

Since the perpendiculars are the co-ordinates, say (x, y), of the point referred to the asymptotes as axes, the equation is

$$xy = \tfrac{1}{2}a^2,$$

or $$xy = K.$$

Curves with this equation occur very frequently in Physics.

7. Normals.

The normal to a conic at any point is the straight line at right angles to the tangent at that point. If the gradient of the tangent at the point x_1y_1 is $g = \left(\dfrac{dy}{dx}\right)_1$, then the gradient of the normal at that point is $-\dfrac{1}{g} = -\left(\dfrac{dx}{dy}\right)_1$.

The equation to the normal is therefore

$$y - y_1 = -\left(\frac{dx}{dy}\right)_1 (x - x_1).$$

This becomes, for

(i) The parabola, $y^2 = 4ax$, $\qquad y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

(ii) The ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\qquad y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$.

(iii) The hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\qquad y - y_1 = -\frac{a^2 y_1}{b^2 x_1}(x - x_1)$.

EXERCISE.—Find the equations to the normals and tangents at the point $x_1 y_1$ on each of the following conics:

(i) $y^2 = 36x$; (ii) $\frac{x^2}{25} + \frac{y^2}{16} = 1$; (iii) $\frac{x^2}{25} - \frac{y^2}{16} = 1$,

and, if possible, at the points for which $x = 3$ and $x = 6$ respectively.

Sub-tangent and Sub-normal.

The length of these is of some importance.

Referring to fig. 9, for point $P(x_1 y_1)$ on the conic, TP is the tangent and RP the normal cutting the axis of x at T and R respectively. PN is the ordinate of P.

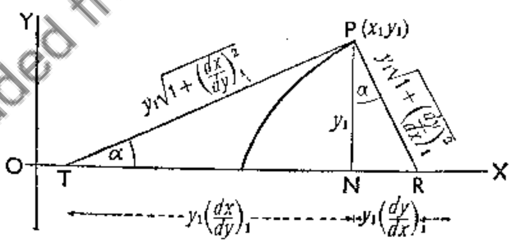

Fig. 9

Then TN is the sub-tangent and RN the sub-normal.

If α is the angle the tangent makes with the axis of x, then if $\left(\frac{dy}{dx}\right)_1$ is the value of $\frac{dy}{dx}$ at P,

$$\left(\frac{dy}{dx}\right)_1 = \tan\alpha.$$

It follows, since $TN = \frac{PN}{\tan\alpha}$, and $RN = PN\tan\alpha$, that,

(i) Sub-tangent $TN = \frac{y_1}{(dy/dx)_1} = y_1\left(\frac{dx}{dy}\right)_1$.

(ii) Sub-normal $RN = y_1\left(\frac{dy}{dx}\right)_1$.

(iii) Tangent $TP = \sqrt{PN^2 + TN^2} = y_1\sqrt{1 + \left(\frac{dx}{dy}\right)_1^2}$.

(iv) Normal $PR = \sqrt{PN^2 + RN^2} = y_1\sqrt{1 + \left(\frac{dy}{dx}\right)_1^2}$.

Example.—For the parabola $y^2 = 4ax$, $\frac{dx}{dy} = \frac{y}{2a}$ and $\frac{dy}{dx} = \frac{2a}{y}$, the sub-tangent is $\frac{y_1^2}{2a} = 2x_1$, and the sub-normal $y_1 \times \frac{2a}{y_1} = 2a$ (constant).

8. Summary.

The equations of all the conics are included in the general form,

$$dx^2 + ey^2 + fxy + gx + hy + k = 0.$$

Thus:

1. If d, f, h and k each $= 0$, and $e = 1$ and $g = -4a$, then

$$y^2 = 4ax. \quad \text{(parabola)}$$

2. If $d = \frac{1}{a^2}$, $e = \frac{1}{b^2}$, $f = 0$, $g = 0$, $h = 0$ and $k = -1$, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \text{(ellipse)}$$

3. If $d = \frac{1}{a^2}$, $e = -\frac{1}{b^2}$, $f = 0$, $g = 0$, $h = 0$ and $k = -1$, then

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad \text{(hyperbola)}$$

4. If d, e, g, h each $= 0$, and $f = 1$ and $k = -K$, then

$$xy = K. \quad \text{(rectangular hyperbola)}$$

5. If d, e, f each $= 0$, then

$$gx + hy + k = 0. \quad \text{(straight line)}$$

9. Polar Co-ordinates.

The position of a point can be defined in terms of its straight-line distance from a fixed point and the direction of this straight line relative to a standard direction.

Thus, if O is a fixed point and OX a standard direction, the position of the point P is given by the length of OP (r) and the angle (θ) which OP makes with OX (fig. 10).

The values of r and θ are called the **polar co-ordinates** of P, O being the **pole**. OP is sometimes called a **radius vector.**

The three conics can be expressed by equations, called **polar equations,** which give the relation between r and θ for all points on the curves.

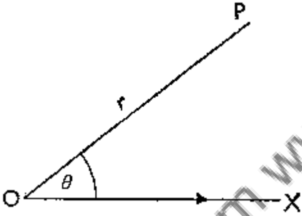

Fig. 10

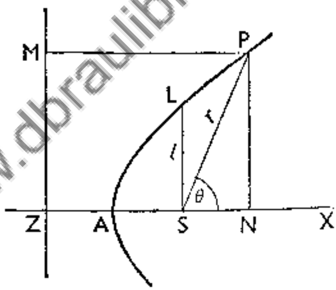

Fig. 11

If ZM is the directrix, S the focus, SL the semi-latus-rectum (l), and e the eccentricity of the conic, then taking S as the pole and (r, θ) as the polar co-ordinates of any point P on the curve, ZX being the standard direction, we have (fig. 11)

$$r = \text{SP} = e\text{PM} = e\text{NZ} = e(\text{ZS} + \text{SN}) = e\text{ZS} + e\text{SN},$$

and since $l = e\,.\,\text{ZS}$ and $\text{SN} = r\cos\theta$,

$$r = l + er\cos\theta \quad \text{or} \quad r = \frac{l}{1 - e\cos\theta}.$$

As before, the conic is a *parabola* when $e = 1$, an *ellipse* when $e < 1$, and a *hyperbola* when $e > 1$.

The reader should trace the curves, taking the same values of e as before, viz. 1, $\frac{3}{5}$, $\frac{5}{3}$, the value of l from each of the conics in fig. 1, and calculating r for convenient values of θ from 0° to 360°.

Exercise XXIX (B)

1. Find expressions for the sub-tangent and sub-normal of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, for any point $P(x_1y_1)$.

2. Show that the equation to the tangent at any point (x_1y_1) of the rectangular hyperbola $xy = k$ is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1),$$

which reduces to

$$x_1y + xy_1 = 2k,$$

and that x_1y_1 is the middle point of the tangent terminated by the axes of x and y. Show also that the area of the triangle formed by the tangent and the intercepts of the axes is $2k$ and therefore constant.

3. Find the co-ordinates of the points of intersection of the parabola $y^2 = 27x$ and the hyperbola $y^2 = 9(x^2 - 4)$.

4. Show that the tangent to the parabola $y^2 = 4ax$ at x_1y_1 cuts the axis of x at $x = -x_1$, and state the method of drawing a tangent based on this fact.

5. Find the angles between the tangents to the conic $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the points for which $x = \pm 3$.

6. The volume of revolution generated by a curve is $\pi\int y^2 dx$ (Chap. XXVIII, 8). Show that the volume generated by

   (i) the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is

$$\frac{\pi b^2 x}{3a^2}(3a^2 - x^2);$$

   (ii) the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is

$$\frac{\pi b^2 x}{3a^2}(x^2 - 3a^2).$$

7. Prove that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

8. Referring to equation (iii) (p. 357), find the values of $\cosh x$ and $\sinh x$ for various values of x, say from 0 to 2·5, then plot the values of $\cosh x$ on the axis of x, and the corresponding values of $\sinh x$ on the axis of y, and verify that the graph is a rectangular hyperbola.

Revision Exercise IV

1. Find the great circle distance between Bermuda (32° 19′ N., 64° 50′ W.) and Plymouth (50° 22′ N., 4° 7′ W.). (The Prime Minister's aerial journey, January, 1942.)
2. (i) Find $e^{-x/4} \sin x$ when $x = \pi/4$.

   (ii) Plot the graph of $e^{-x/4} \sin x$ for two cycles and notice the effect of $e^{-x/4}$ on the graph of $\sin x$.
3. A telephone current diminishes in the ratio e^{-cx} in a distance x miles. Find the current at a point 100 miles from the transmitting station, if at this station the current is 1 unit, and if c is 0·0125.
4. When a rope is wrapped round a round post and a pound weight attached to one end, it is found that the pull P, to be applied at the other end to cause the rope to slip, is given by the equation

$$P = e^{\mu\theta} \text{ (lb.)},$$

in which μ is the coefficient of friction between the rope and the post, and θ is the angle of lap round the post in radians.

Find P when the rope is lapped twice round and the coefficient μ is 0·25.

5. If $x = \frac{E}{R} + Ce^{-\frac{Rt}{L}}$, and $x = 0$ when $t = 0$, find C. Then find x when $E = 100$, $R = 20$, $L = 10$ and $t = 0{\cdot}05$.
6. (i) Putting $z = nx$, show that $\frac{d}{dx}(\sin nx) = n \cos nx$.

   (ii) Putting $z = \sin x$, show that $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$.
7. If $y = \sqrt{a^2 - x^2}$, find $\frac{dy}{dx}$ (substitute $z = a^2 - x^2$).
8. If $z = \sin x$, then $\frac{dz}{dx} = \cos x$, and $\int \sin^n x \cos x \, dx = \int z^n \, dz$. Show that the integral equals $\frac{\sin^{(n+1)} x}{n+1}$.
9. Evaluate $\int \sin^2 x \, dx$, expressing $\sin^2 x$ in terms of $2x$.
10. Show that $y = k \cos(nx + C)$ satisfies the equation $\frac{d^2y}{dx^2} + n^2 y = 0$.
11. The equation $y = \frac{c}{2}(e^{x/c} + e^{-x/c})$ is that of a catenary (the form taken by a hanging flexible wire supported at each end). Show that near the vertex the catenary nearly coincides with the parabola $y = c + \frac{x^2}{2c}$.

12. Find the points of intersection of the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ and the parabola $4y^2=27x$ and the volume of revolution generated when the area between the curves revolves about the axis of x.

13. Find the equations of the tangents to the curves of Ex. 12 at one of the points of intersection, and the angle between the tangents.

14. A radius, the length (r) of which varies according to the law $r=a+b\cos\theta$, rotates about one end, θ being the angle of rotation. The area swept out by r is $\frac{1}{2}\int r^2 d\theta$. Find the area for one complete cycle.

15. A barrel having the shape of the volume of revolution generated by the revolution of part of an ellipse about its major axis, and bounded by plane ends, has the following dimensions: mid-diameter, 3 ft.; length, 6 ft.; diameter of ends, 2 ft. Find its volume.

16. Draw the graph of $y=\sqrt{a^2-x^2}$ from $x=0$ to $x=a$. Draw any ordinate for a value of x between 0 and a, and show that the area between this ordinate and that at $x=0$ is equal to $\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$. Show also that

$$\text{(i)}\quad \int\sqrt{a^2-x^2}\,dx=\tfrac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a},$$

and

$$\text{(ii)}\quad \int_0^a\sqrt{a^2-x^2}\,dx=\frac{\pi a^2}{4}.$$

SUMMARY OF IMPORTANT RELATIONS

Algebra.

$+ \times + = +, \quad - \times - = +, \quad + \times - = -.$

$x + x + x + \ldots (n \text{ terms}) = nx.$

$x \times x \times x \times \ldots (n \text{ factors}) = x^n.$

$(x \pm y)^2 = x^2 \pm 2xy + y^2, \quad (x + y)(x - y) = x^2 - y^2.$

$$(x \pm y)^n = x^n \pm nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 \pm \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots * y^n.$$

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \ldots + y^{n-1}.$$

$$n \text{ even}, \frac{x^n - y^n}{x + y} = x^{n-1} - x^{n-2}y + \ldots - y^{n-1}.$$

$$n \text{ odd}, \frac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + \ldots + y^{n-1}.$$

$$\text{If } ax^2 + bx + c = 0, \; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$a + (a + d) + (a + 2d) + \ldots + (a + \overline{n-1}d) = \tfrac{1}{2}n(2a + \overline{n-1}d).$$

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r}.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \; .$$

$\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \sinh x = \frac{1}{2}(e^x - e^{-x}).$

$\cosh x \pm \sinh x = e^{\pm x}, \quad \cosh^2 x - \sinh^2 x = 1.$

$(\cos x \pm i \sin x)^n = e^{\pm inx} = \cos nx \pm i \sin nx.$

Mensuration—

circle: circumference, $2\pi r$; area, πr^2.

cylinder: surface $(2\pi rh + 2\pi r^2)$; vol., $\pi r^2 h$.

cone: surface $(\pi r\sqrt{h^2 + r^2} + \pi r^2)$; vol., $\frac{1}{3}\pi r^2 h$.

sphere: surface $4\pi r^2$; vol., $\frac{4}{3}\pi r^3$.

* For sign, see text p. 293 or 326.

Trigonometry.

Plane—

$\tan A = \frac{\sin A}{\cos A}$, $\sin\left(\frac{\pi}{2} - A\right) = \cos A$, $\cos\left(\frac{\pi}{2} - A\right) = \sin A$.

$\sin^2 A + \cos^2 A = 1$, $\sec^2 A - \tan^2 A = 1$, $\operatorname{cosec}^2 A - \cot^2 A = 1$.

$\sin(-A) = -\sin A$, $\cos(-A) = \cos A$, $\tan(-A) = -\tan A$.

$\sin\left(\frac{\pi}{2} + A\right) = \cos A$, $\cos\left(\frac{\pi}{2} + A\right) = -\sin A$.

$\sin(\pi - A) = \sin A$, $\cos(\pi - A) = -\cos A$.

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.

$\sin 2A = 2 \sin A \cos A$, $\cos 2A = \cos^2 A - \sin^2 A$,

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

$\sin\frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}}$, $\cos\frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}$,

$$\tan\frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}}.$$

$\sin A + \sin B = 2 \sin\frac{A + B}{2} \cos\frac{A - B}{2}$.

$\sin A - \sin B = 2 \cos\frac{A + B}{2} \sin\frac{A - B}{2}$.

$\cos A + \cos B = 2 \cos\frac{A + B}{2} \cos\frac{A - B}{2}$.

$\cos A - \cos B = -2 \sin\frac{A + B}{2} \sin\frac{A - B}{2}$.

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, $a^2 = b^2 + c^2 - 2bc \cos A$.

$\sin\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}$, $\cos\frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}$,

$$\tan\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}.$$

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
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11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
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31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
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35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
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49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
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52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
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											Mean Differences								
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56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
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59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
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70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
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80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
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83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											Mean Differences								
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences 1′	2′	3′	4′	5′
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10°	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20°	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30°	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40°	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences 1′	2′	3′	4′	5′
45°	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50°	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60°	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70°	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80°	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0

NATURAL COSINES

Subtract Mean Differences

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences 1′	2′	3′	4′	5′
0°	1·000	1·000	1·000	1·000	1·000	1·000	·9999	9999	9999	9999	0	0	0	0	0
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10°	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20°	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30°	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40°	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

Subtract Mean Differences

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences 1′	2′	3′	4′	5′
45°	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50°	·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60°	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70°	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80°	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences 1′	2′	3′	4′	5′
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10°	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20°	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30°	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40°	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

	0	6′	12′	18′	24′	30′	36′	42′	48′	54′	Mean Differences 1′	2′	3′	4′	5′
45°	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50°	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60°	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	$\overline{0057}$	$\overline{0145}$	$\overline{0233}$	$\overline{0323}$	$\overline{0416}$	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70°	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	$\overline{0061}$	$\overline{0237}$	$\overline{0415}$	$\overline{0595}$	29	58	87	116	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	4·7046	7453	7867	8288	8716	9152	9594	$\overline{0045}$	$\overline{0504}$	$\overline{0970}$					
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80°	5·6713	7297	7894	8502	9124	9758	$\overline{0405}$	$\overline{1066}$	$\overline{1742}$	$\overline{2432}$					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	$\overline{0264}$	*Mean* differences no longer sufficiently accurate.				
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	$\overline{0285}$					
83	8·1443	2636	3863	5126	6427	7769	9152	$\overline{0579}$	$\overline{2052}$	$\overline{3572}$					
84	9·514	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

ANSWERS

Ex. I (A)

1. 105, 105, 106, 105, 106, 105. **2.** 39·37, 0·29, 0·50, 0·49.

Ex. I (B)

1. (i) 135·6, (ii) 3·87, (iii) ·025, (iv) 2200, (v) 220,000, (vi) 30,000, (vii) ·05, (viii) 325, (ix) 1·5385, (x) 1,250,000, (xi) 25·64, (xii) 4,800, (xiii) ·0675.
2. 39·6. **3.** ·6214. **5.** 1 hectare = $2\frac{1}{2}$ acres (approx.).

Ex. I (C)

1. ·05, ·04. **2.** (i) ·1, (ii) ·94, (iii) ·58, (iv) ·3. **3.** 16·39.
4. 19·58. **5.** 51·194. **6.** 2·501. **7.** 13·19. **8.** ·6736.
9. ·5064. **10.** 3·1424. **11.** 7·6812. **12.** 5·184.

Ex. II (A)

1. (i) 90, 45, 135, 225, 225, 135, 135, 90, degrees; (ii) 270, 315, 225, 135, 135, 225, 225, 270, degrees. **3.** (i) 90°, (ii) 360°. **7.** 20°.

Ex. II (B)

3. Right angles.

Ex. III (A)

1. +£250 as contrasted with −£250. **2.** −£60. **3.** £60.
4. £115. **5.** −£115. **6.** Assets, £850; debts, £500; state, £350.
7. Zero. **8.** −£150.

Ex. III (B)

1. (*a*) 12°; (*b*) −5°; (*c*) 0°; (*d*) 100°; (*e*) −15°. **2.** 25°. **3.** 10°.
4. −10°. **5.** 15°. **6.** 10. **7.** 20, −5, 0, 3, 5. **8.** 180. **9.** 190.

Ex. III (C)

2. +2·3 cm. **3.** −7·9 cm. **4.** −2·6 cm.

Ex. III (D)

1. 60°. **2.** +20°. **4.** +3028 ft., −900 ft.
5. Clockwise, +60 r.p.m., −60. r.p.m.
6. −100 r.p.m., −200 r.p.m. if belt uncrossed; −100 r.p.m., +200 r.p.m. if crossed.
7. +15 min., −10 min. **8.** 8 min. per day, −5 min. per day.
9. +32·2 ft. per sec. each sec., −32·2 ft. per sec. each sec.
10. +12 lb., −12 lb. **11.** By using plus and minus signs.
12. −55, +1915.

Ex. IV (A)

1. 21. **2.** −11. **3.** −5. **4.** 5. **5.** 7.
6. 0. **7.** 13, −5. **8.** −13. **9.** 3. **10.** −3.

Ex. IV (B)

1. 20. **2.** −20. **3.** 4. **4.** −4. **5.** −25. **6.** −15.
7. 15. **8.** 25. **10.** 16. **11.** −19. **12.** $12\frac{1}{4}$ miles W.

Ex. IV (C)

1. 4. **2.** −4. **3.** 16. **4.** −16. **5.** 4. **6.** −4. **7.** 16. **8.** −16.
9–12. Regarded from the second number, −28, −8, 8, 28. **13.** −6.
14. 0. **15.** −4. **16.** −10. **17.** 10. **18.** 10. **19.** 13. **20.** 13. **21.** 3
22. 5. **23.** −5. **24.** 14. **25.** 1. **26.** 9. **27.** 20. **28.** −4.

Ex. IV (D)

1. 18. **2.** −18. **3.** −18. **4.** 18. **5.** 1. **6.** −1. **7.** −1.
8. 1. **9.** $-\frac{3}{8}$. **10.** 1. **11.** 0. **12.** 0. **13.** 0. **14.** −16.
15. −90. **16.** 60. **17.** −36. **18.** 4. **19.** −8. **20.** −810.
21. +5 × +3 = +15, +5 × −3 = −15, −5 × +3 = −15, −5 × −3 = +15.
22. (i) minus, (ii) plus.

Ex. IV (E)

1. 5. **2.** −5. **3.** −5. **4.** 5. **5.** $\frac{1}{4}$. **6.** $-\frac{1}{5}$.
7. $-\frac{1}{5}$. **8.** $\frac{1}{8}$. **9.** −6. **10.** 6. **11.** 3. **12.** −3.
13. −2. **14.** 0. **15.** 5. **16.** 5. **17.** 0. **18.** −5.
19. −7. **20.** 4. **21.** −4. **22.** 12. **24.** (i) 6, (ii) $-1\frac{1}{3}$.

Ex. V (A)

1. $7a$. **2.** $8x$. **3.** $3x$. **4.** $9a$. **5.** $8y$. **6.** $2a + 9c$.
7. $7x$. **8.** $-4x + 5$. **9.** $8p$. **10.** $2p$. **11.** $-8p$. **12.** $a + 3b$.

13. $3a + 2b$. **14.** $-5x + 2y$. **15.** $5x - 2y$. **16.** $3x$. **17.** $(w + x)$ gm. **18.** $(x + 100)$ gm. **20.** $8a + 11b + 6c$. **21.** $-b + 4c$. **22.** $2x + a + 5b$, 18. **23.** $4a - 3c = -5$; lines, $+14, -2, -10, 0, -7$. **24.** $8a + 7x - 8y$. **25.** (i) 0, (ii) 12, (iii) -5, (iv) -13, (v) 1, (vi) 5, (vii) $3a + 13$, (viii) $c - 1\frac{7}{12}$. **26.** (i) 7, (ii) -3. **27.** (i) 0, (ii) $6a$, $6b$, or $6c$, (iii) $1\frac{3}{5}a$, $1\frac{3}{5}b$, or $1\frac{3}{5}c$.

Ex. V (B)

1. (i) $4a$, (ii) $-4a$, (iii) $8a$, (iv) $-8a$, (v) $-4a$, (vi) $4a$, (vii) $8a$, (viii) $-8a$.
3. $a - 7b + 8c$. **4.** (i) $2x - 2$, (ii) x, (iii) $-x$, (iv) $2x$.
5. $-a - 7b - 2$, -17. **6.** $-2a + 7b - 4c$, 29.
7. Regarded from the first term, (i) $2x$, (ii) a, (iii) $3x - 3$, (iv) $x + y$. Regarded from the second term, (i) $-2x$, (ii) $-a$, (iii) $3 - 3x$, (iv) $-x - y$.
8. $(y - x)$, $(z - y)$ gm. **9.** $(w + m + n - x)$ gm., $(w + m + n - x)$ gm.
10. (C − W), (C − M), (M − W) gm.

Ex. V (C)

1. ax. **2.** ax. **3.** abx. **4.** $-6ax$. **5.** $-2xy$.
7. $ax + 2a$. **8.** $-3x - 6$. **9.** $3ax + 6a$, $-3ax - 6a$.
10. $-6ax + 4bx$. **12.** a^2, $-a^2$, $-a^2$, a^2.
14. (i) $-6x^3y^3$, (ii) $3ab^3$, (iii) $4x^2$, (iv) $-4x^2$, (v) $4x^2$, (vi) x^6, (vii) $9a^4b^2$, (viii) $-a^3$, (ix) $8a^6$. (i) -1296, (ii) 9, (iii) 16, (iv) -16, (v) 16, (vi) 64, (vii) 729, (viii) 27, (ix) 5832.
15. $\frac{1}{2}xy$, xy, xy, $\frac{3}{2}bh$, $3b$.

Ex. V (D)

1. $-a$. **2.** a. **3.** $3x^3$. **4.** $-\dfrac{1}{3x^4}$. **5.** $-3a^2$. **6.** $\frac{3}{2}b^3$.
7. $3a^2b$. **8.** $-4xy$. **9.** $a + 4b$. **10.** $x - y^2$. **11.** $10x$.
12. $-\frac{1}{2}a^2$. **13.** $\dfrac{x + y^2}{2}$ or $\frac{1}{2}x + \frac{1}{2}y^2$. **14.** $x - 3xy^2$. **15.** $5a^2b^2 + b - 4$.
16. $-4y^3$. **17.** $x^2 + 5x - 4$. **18.** $\dfrac{6}{y}$. **19.** $\dfrac{5}{y}$. **20.** $\dfrac{6}{x}$.
21. (i) $-1\frac{1}{3}$, (ii) $-\frac{2}{3}$, (iii) -54, (iv) $4\frac{7}{36}$, (v) 6, (vi) 6.
22. $\dfrac{1}{x^3}$. **23.** $\dfrac{a - b}{a + b}$. **24.** 228.

Ex. V (E)

1. 1, 1, $4x^2$, $9x^2$, $25x^4$, $9x^6$, $16x$, $-x$, x^3, $9(a + b)^2$.
2. $\pm 5a$, $\pm 3b^2$, $\pm 7b^6$, $\pm 8y^5$, $\pm 5(a + b)$. **4.** $\pm \dfrac{x}{2y^2}$, $\pm \dfrac{x}{4y^4}$, $\pm \dfrac{1}{x}$, $\pm \dfrac{1}{x}$, $\pm 3x$.
5. ± 4, ± 5, ± 8, ± 10, ± 12, $2\sqrt{3}$, $2\sqrt{5}$, $2\sqrt{7}$, $6\sqrt{2}$, $4\sqrt{2}$, $-5\sqrt{3}$, ± 12, $12\sqrt{2}$, $2x\sqrt{2}$, $2x\sqrt{2}$, $4x^2y\sqrt{2}$, $x\sqrt{2}$.
7. 10^3, 3; 10^2, 2; 10^0, 0; 10^4, 4; 10^6, 6; 10^7, 7.
8. 2, 5, 6, 3, 3. **9.** 2^5, 2^6, 3^5, 5^{10}. **10.** 5.

Ex. VI (A)

5. 150 sovs. and £40 debts. 6. $10x - 15y$. 7. $-10x + 15y$.
8. $2x^2 - 3xy$. 9. $-2x^2 + 3xy$. 10. $-2xy - 3y^2$. 11. $-2xy + 3y^2$.
12. $a^2 + b^2 + c^2$. 13. $2a - 8b + 3c$. 14. $3x^2 - 2xy - 3x - 14y$. 15. $12a - 3b$.
16. $-4a + 6b$. 17. 0. 18. $2(x + y)$. 19. $-2(x + y)$.
20. $-2(x - 3y)$. 21. $3(a - 2b + 4c)$. 22. $3(a - 2b) + 5(c + 5d)$.
23. $3(a - 2b) - 5(c - 5d)$. 24. $a(a - b) - c(c - d)$. 25. $a(a + b) - b(a - b)$.
26. $x(x^2 - 3xy - y^2)$. 28. $x - 2y$. 29. $3x^2 + 4y^2$. 30. $2x - 5y$.

Ex. VI (B)

1. $-13x - 3y$. 2. $-a - ax + 5ay + 4b$. 3. $2a - 3$.
4. $-34x + 48$. 5. 27. 6. $13x - y - 4$.
7. $2a\{3a - b(a - b)\}$. 8. $a\{a(x^2 - y^2) + xy\}$.
9. $a\{b - c(b - 1)\}$. 10. $x(ax + b) - y(cy - d)$.
11. $a(x + y + c)$. 12. $(a + b)(a - b)$.
13. $(p + q)(x + y + c)$. 14. $(p + q)(x + 2y + z)$.
15. $2y(p + q)$. 16. $(a + b)(a + b)$.
17. $2(a + b)$, 12·2 in. 18. $2\{h(a + b) + ab\}$.
19. 205·19. 20. 3·8104.

Ex. VI (C)

1. x^2y, x^3y^2. 2. ab^2c^2, $a^3b^3c^4$. 3. a^2, a^4b^3.
4. $5x^3y^3$, $30x^4y^4$. 5. $3abc^2$, $6a^2b^2c^3$. 6. $a^2b^3c^3$, $a^5b^4c^6$.
7. 1, $6abc$. 8. 1, $6ac$. 9. 5, $60abc$.
10. $a + b$. 11. $5 + 3d$. 12. $\frac{1}{3}$.
13. $\frac{a + 2}{x}$. 14. $\frac{2a + b}{2x}$. 15. $\frac{a^2 + x^2}{ax}$.
16. $-\frac{1}{2}$. 17. $\frac{3y}{2(a - b)}$. 18. $\frac{a}{c}$.
19. $-\frac{b}{c}$. 20. $\frac{x^2 + 2y^2}{3xy}$. 21. $\frac{3x^2 + 4y^2}{6xy}$.
22. $-\frac{5a + 7}{12}$. 23. $\frac{9y - 7x - 20}{10}$. 24. $\frac{2a^2}{3b}$.
25. $\frac{3x}{ay^2}$. 26. 45. 27. $\frac{b(ax + 1)}{a(bx + 1)}$.
28. $\frac{a}{b}$. 29. $ab(a + b)$. 30. $6ab(a + 1)$.

Ex. VII (A)

3. 60°. 4. 54°. 5. 30°, 60°, 90°.

Ex. VII (B)

2. 2·6 in., 41°, 79°.

Ex. VII (G)

2. $2\frac{1}{2}$ in. **7.** $10\sqrt{3}$, 20 cm. **8.** 5·3 cm. nearly.

Ex. VII (I)

1. (i) 31·4 cm., (ii) 33 cm., (iii) 88 in., (iv) $6\frac{2}{7}$ ft., (v) 22 in., (vi) 11 ft.
2. (i) 78·5 sq. cm., (ii) $86\frac{5}{8}$ sq. in., (iii) 616 sq. in., (iv) $3\frac{1}{7}$ sq. ft., (v) $38\frac{1}{2}$ sq. cm., (vi) $9\frac{5}{8}$ sq. ft.
3. π in. **4.** 14 in. **5.** 7955 miles. **6.** 154 sq. in.
7. 25·7 cm. **8.** 7 in., 77 sq. in. **9.** πr^2 or $\frac{1}{2}cr$.
10. (i) $a + b + c + d$, (ii) $4a$, (iii) $2a + 2b$, (iv) $2a + 2b$, (v) $4a$, (vi) $a + b + c$, (vii) $2a + b$, (viii) $3a$.
11. $\pi r + 2r$, $\frac{1}{2}\pi r + 2r$. **15.** $r^2 - \frac{1}{4}\pi r^2$, $\frac{1}{4}\pi r^2 - \frac{1}{2}r^2$, $\frac{1}{2}\pi r^2 - r^2$.
16. lbh, $\frac{1}{2}abh$, $\pi r^2 h$. **17.** $\frac{1}{3}abh$, $\frac{1}{6}abh$, $\frac{1}{3}\pi r^2 h$.

Ex. VIII (A)

1. 4.	**2.** 9.	**3.** −3.	**4.** 0.	**5.** 0.
6. −10.	**7.** 5.	**8.** −5.	**9.** −5.	**10.** −3.
11. −3.	**12.** 3.	**13.** $3\frac{1}{3}$.	**14.** $-3\frac{1}{3}$.	**15.** $-3\frac{1}{3}$.
16. $3\frac{1}{3}$.	**17.** $\frac{1}{3}$.	**18.** $-\frac{1}{3}$.	**19.** $\frac{1}{3}$.	**20.** $-\frac{1}{3}$.
21. −3.	**22.** −3.	**23.** −3.	**24.** 5.	**25.** 5.
26. 1.	**27.** 3.			

Ex. VIII (B)

1. 6.	**2.** 3.	**3.** −3.	**4.** 3.	**5.** $\frac{1}{2}$.	**6.** 4.
7. −4.	**8.** −4.	**9.** 12.	**10.** 3.	**11.** −3.	**12.** $\frac{1}{3}$.
13. −1.	**14.** $\frac{3}{2}$.	**15.** 1.	**16.** $2\frac{7}{9}$.	**17.** $\frac{9}{35}$.	**18.** −3.
19. 10.	**20.** −1.	**21.** $20\frac{1}{6}$.	**22.** 21.	**23.** 5.	**24.** 60.
25. 135.	**26.** $20\frac{1}{2}$.	**27.** $26\frac{2}{3}$.	**28.** −3.	**29.** 2.	**30.** 3.

31. £11. 17s. 6d.
32. (i) £3·176 + ·01d., (ii) £3·377 + ·02d., (iii) £3·877 + ·02d., (iv) £3·384 + ·09d. (i) £635. 4s. 2d., (ii) £755. 8s. 4d., (iii) £775. 8s. 4d., (iv) £676. 17s. 6d.
33. 11 a.m., 5 a.m., 7 a.m., 5 p.m., 11.20 p.m.

Ex. IX (A)

1. $ac - ad + bc - bd$. **2.** $ac + ad - bc - bd$.
3. $6ac + 4ad + 9bc + 6bd$. **4.** $6ac - 4ad + 9bc - 6bd$.
5. $6ac - 4ad - 9bc + 6bd$. **6.** $6m^2p^3 - 9m^2q^3 - 10n^2p^3 + 15n^2q^3$.

7. $x^2 - 5x + 6$. 8. $x^2 + 5x + 6$. 9. $x^2 - x - 6$.
10. $x^2 + x - 6$. 11. $4x^2 - 4xy - 3y^2$. 12. $12x^2 - 26x + 12$.
13. $ax - a^2$. 14. $x^3 - a^2x$. 15. $ac + ad + ae + bc + bd + be$.
16. $a^2 - b^2 + ac + bc$. 17. $ab - 4a + 3b - 12$.
18. $10x^2y^2 - 19xy - 15$. 19. $x^3 + x^2 - 7x + 2$.
20. $2a^4 - 7a^3 + 10a^2 - 7a + 2$. 21. $x^4 + x^3y - 13x^2y^2 - xy^3 + 12y^4$.
22. $x^3 - \frac{1}{x^3}$. 23. $a^3 - b^3 + c^3 + 3abc$.
24. $4 + 4x - 9x^2 + 5x^3 - 15x^4 + 14x^5 - 6x^6$.

Ex. IX (B)

1. $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$, $a^2 - b^2$.
2. $4a^2 + 4ab + b^2$, $4a^2 - 4ab + b^2$, $4a^2 - b^2$.
3. $a^2 + 4ab + 4b^2$, $a^2 - 4ab + 4b^2$, $a^2 - 4b^2$.
4. $4x^2 + 20x + 25$, $4x^2 - 20x + 25$, $4x^2 - 25$.
5. $\frac{x^2}{4} + x + 1$, $\frac{x^2}{4} - x + 1$, $\frac{x^2}{4} - 1$.
6. $4x^2 + 12xy + 9y^2$, $4x^2 - 12xy + 9y^2$, $4x^2 - 9y^2$. 7. 289, 2209, 2475.
8. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$, $a^2 + b^2 + c^2 + 2ab - 2bc - 2ac$, $a^2 + 2ab + b^2 - c^2$.
9. $4a^2 + b^2 + 4c^2 - 4ab - 4bc + 8ac$, $4a^2 + b^2 + 4c^2 - 4ab + 4bc - 8ac$, $4a^2 + b^2 + 4c^2 - 4ab$.
10. $144a^2 + 144ab + 36b^2$, $a^4 - 2a^2b^2 + b^4$.

Ex. IX (C)

2. $a + b$. 3. $a - b$. 4. $a - b$. 5. $a + b$. 6. $a - b$, rem. $2b^2$.
7. $a + b$, rem. $2b^2$. 8. $x - 2$. 9. $a^2 - 2a + 2$. 10. $3x^2y^2 + xy - 4$.
11. $x^2 + x + 3$. 12. $a^3 - 2a^2b + 4ab^2 - 8b^3$. 13. $3 + a - a^2$.
14. $2x^2 + xy - 3y^2$. 15. $x^2 + 4x + 8$. 16. $1 + 2a + a^2$.
17. $2x - 2$, $x - 1$. 18. 6.

Ex. IX (D)

1. (i) b^2, (ii) b^2, (iii) $\pm 2ab$. 2. (i) $4y^2$, (ii) $4y^2$, (iii) $\pm 4xy$.
3. (i) a^2, (ii) a^2, (iii) 1. 4. (i) x^2, (ii) y^2, (iii) 1.
5. (i) $4y^2$, (ii) $4y^2$, (iii) 4. 6. (i) $\frac{1}{4}$, (ii) $\frac{1}{4}$, (iii) ·16. 7. $25y^4$.
8. 9. 9. $9b^2y^2$. 10. y^2. 11. b^2. 12. $\frac{4}{y^2}$.

Ex. IX (E)

1. $2a + 5b$. 2. $2a - 5b$. 3. $4x - 5y$. 4. $6x^2 - 1$.
5. $a^2 - 2a - 2$. 6. $1 - 2y + y^2$. 7. $x^2 - xy + 2y^2$. 8. $3a^2 - 2a + 1$.
9. 99. 10. 123. 11. 55·5. 12. 1·414.
13. 1·732. 14. 2·236. 15. 2·449. 16. 2·646.

17. 2·875. **18.** ·196. **19.** 15·215. **20.** 30·795.
21. (i) 13 cm., (ii) 25 cm., (iii) 41 cm., (iv) 85 in., (v) 65 ft., (vi) 37 cm., (vii) 4·3 in.
22. (i) 12 cm., (ii) 55 cm., (iii) 108 cm., (iv) 20 in., (v) 60 in., (vi) 77 ft., (vii) 3 cm., (viii) 27·7 ft.
23. 19·4 ft. **24.** 11·31 cm. **25.** 18·03 cm.

Ex. X (A)

1. 90°, 108°, 120°, $128\frac{4}{7}$°, 135°, 140°, 144°. **2.** 180°, circle.

Ex. XI (A)

1. $\frac{5}{8}, \frac{2}{3}, \frac{5}{3}, \frac{3}{4}, \frac{35}{44}, \frac{3}{5}, \frac{2}{3}, \frac{5}{8}$. **3.** 4 : 1. **4.** 4 : 1. **5.** 1 : 12, no.

6. 4 : 5. **7.** 22 : 7, 11 : 14 (approx.). **8.** $\frac{4}{16} - \frac{\pi}{} = \frac{3}{56}$ (approx.).

9. $\frac{5}{8}, \frac{5}{6}, \frac{6}{11}, \frac{1}{2}, \frac{3}{7}, \frac{2}{5}$. **10.** $\frac{M - B}{W - B}$.

Ex. XI (B)

1. $8\frac{1}{3}$. **2.** $2\frac{3}{5}$. **3.** $8\frac{3}{4}$. **4.** $22\frac{3}{5}$. **5.** $\frac{ab}{c}$. **6.** $\frac{ac}{b}$. **7.** $\frac{a^2}{b}$. **8.** b.
9. (i) directly, (ii) inversely, (iii) directly, (iv) inversely.
10. (i) 3, (ii) $8\frac{1}{3}$, (iii) 12, (iv) $22\frac{1}{2}$. **11.** (i) inversely, (ii) directly.
12. 32,800 sq. miles. **13.** 2·618 cm. **14.** $\frac{1}{6}$.
15. $\frac{x}{360}, \frac{\pi R^2 x}{360}$. **16.** Ratios inversely equal. **17.** 128·3 gm.
18. 7·854 in. **19.** 57·3°, 28·65°. **20.** $\frac{11x}{12}$. **21.** $\frac{A}{A - W}$.
22. $\frac{A - T}{A - W}$. **25.** 4, inversely. **27.** 1st : 2nd = 3 : 1.

Ex. XI (C)

6. 56′ 3″. **7.** 2″. **8.** 13 cm.
10. $5\frac{5}{8}$ and $4\frac{3}{8}$, 9 and 5, $10\frac{1}{2}$ and $7\frac{1}{2}$. **12.** 2 cm.

Ex. XII (A)

1, 2. See tables. **3.** Sin of angle equals cos of complement. **4.** See tables.
5. Side opposite angle nearly 90° is nearly equal to the hypotenuse; side opposite angle nearly 0° is very small.

Ex. XII (B)

1. 5·196, 6 in. **2.** 3·81, 5 cm. **3.** 1·5, 2·598 in.
4. 2·06, 2·45 cm. **5.** 45·9, 45·72 in. **6.** 1·607, 1·915 in.
7. 7·8 sq. in., 6·11 sq. cm., 1·96 sq. in., 2·52 sq. cm., 91·4 sq. in., 1·53 sq. in.
8. 251·7 ft. **9.** 37° (approx.). **10.** 122 yd.

Ex. XII (c)

1. ·35, ·52, ·79, 1·05, 1·57, 2·27, 3·49, 4·71, 5·41, 6·28.
2. 12, 63, 138, 180, $303\frac{1}{2}$, 360, 360, 270.
3. Answers of 1 multiplied by (*a*) 5, (*b*) 10; Radians of 2, multiplied by (*a*) 5, (*b*) 10. **4.** 1·05 in.

Ex. XII (d)

1. 96 sq. in. **2.** $76\frac{1}{2}°$, $76\frac{1}{2}°$.
3. 1072 yd. (approx.). **4.** 7·15 cm., 71·5 sq. cm. (approx).
5. 40 sq. cm., 12 cm. **6.** $47\frac{1}{4}$ sq. in.

Ex. XII (e)

2. 1·48, 2·27 cm. **3.** 1·76, 1·31 cm. **4.** 2·69, 3·06 cm.
5. 7·66, 6·43 cm. **6.** 7·71, 3·92 in. **7.** 2·31, 2·03, 3·11, 10 cm., 7·83 in.
8. (i) 815·3, 921·5 yd.; (ii) 475 ft., 9° 45′.
10. 1·965, 1·13, 2·65, 24·6 sq. cm.; 11·6 sq. in.
11. 15,474 miles, 644·8 miles per hour.
13. 34 ft. **14.** 176·9 yd. **15.** $7\frac{1}{2}$ million sq. miles.
16. $1\frac{1}{2}$ million sq. miles. **17.** 3715 miles. **18.** 4170 miles.
19. 84°, 4·97 ft.; 192°, 168°, 17·9 ft.; 150 r.p.m., 15·7. **20.** 21·29 in.

Revision Ex. I

1. (i) -4, (ii) 6, (iii) $7(a+b)$, (iv) $4a(a-b)-2(x+y)$.
2. (i) -20, (ii) 6, (iii) $-8x^2+8x+3$, (iv) $(a-3b)$. **3.** 0·9 in.
4. (i) 0, (ii) 1, $\frac{1}{4}$; 2, -1. **5.** (i) $s\left(\frac{\pi}{2}+2\right)$, (ii) $\frac{s^2}{8}(\pi+2\sqrt{3})$.
6. 94. **7.** $a^2+4x^2+9y^2+4ax-12xy-6ay$.
8. (i) 21, (ii) -27, No. **9.** (i) -6, (ii) 9.
11. 6·34, 12·68, 10·98 in., 34·8066 sq. in. **12.** (i) 0·183, $\frac{1}{3}$, (ii) 1·618.
13. $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$.

Ex. XIII (a)

1. (i) -5, $1\frac{2}{3}$; (ii) 4, 4; (iii) -7, $10\frac{1}{2}$.
2. (i) $y=3x-7$, (ii) $y=-3x+5$, (iii) $y=5x-3$, (iv) $y=-2x+12$, (v) $y=\frac{2}{3}x$. **3.** $y=5$.
4. $y=\frac{1}{2}x+5$, $y=\frac{1}{2}x-5$, $y=-\frac{1}{2}x+5$, $y=-\frac{1}{2}x-5$, $y=5$, $y=-5$, $x=5$, $x=-5$.
7. The graph becomes steeper, and finally is vertical.
8. $y=-0·1x-3$, $y=0·4x+30·1$.

Ex. XIII (B)

8. $2\frac{3}{4}, -\frac{1}{2}$. **9.** 14th. **10.** 8th.

11. (i) 4·48 p.m., (ii) 4·50 p.m. Thursday, $y = 10x - 30$, $y = 50 - 15x$.

12. $y = 2x + 3$. **13.** $y = \frac{3}{2}x - 6$. **14.** $y = -2x + 3$.

15. $y = \frac{1}{2}x$. **16.** 5. **17.** $y = -x + 3$.

18. 3, 12, −9. **19.** (i) to (v) −3, the coefficient of x in all cases.

Ex. XIII (C)

1. $f = 0{\cdot}8h + 8$. **3.** $f = 0{\cdot}217w + 1{\cdot}5$. **4.** $R = 0{\cdot}073D + 30$.

5. Temp. $= 3t + 15$. **6.** $L = -0{\cdot}687t + 605{\cdot}7$.

7. $R = 0{\cdot}344t + 100$. **8.** $V = 0{\cdot}00128t + 1$.

Ex. XIII (D)

4. $V = \dfrac{450}{P}$. **5.** $D = \dfrac{500}{W}$. **6.** $y = \dfrac{3}{x} + 2$, $x = 0$, $y = 2$.

Ex. XIV (A)

1. $11\frac{1}{6}, 7\frac{4}{5}$. **2.** 3, 2. **3.** 6, 4. **4.** $\frac{5}{6}, -\frac{5}{6}$.

5. $1\frac{8}{19}, 1\frac{7}{19}$. **6.** 5, 7. **7.** $\frac{1}{10}$, 10. **8.** $2, \frac{1}{3}$.

9. 3, 2. **10.** $8\frac{5}{7}, 3\frac{1}{7}, 3$. **11.** $\frac{5}{2}, \frac{1}{2}$. **12.** 20, 20.

13. 17·4, 12. **14.** −2, 1. **15.** 9, 11, 13.

16. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$. **17.** 4, 0, 5. **18.** $7\frac{1}{2}, -4$.

Ex. XIV (B)

1. $a + b + 1$. **2.** $\dfrac{3a^3b^2}{5c}$. **3.** $b - c$. **4.** a, b.

5. 1. **6.** a, b. **8.** $\dfrac{c}{2\pi}$. **9.** $2\sqrt{\dfrac{A}{\pi}}$.

10. $\dfrac{c}{\pi} - b$. **11.** $\dfrac{A}{\pi a}$. **12.** $\dfrac{1}{2}\sqrt{\dfrac{A}{\pi}}$. **13.** $\sqrt[3]{\dfrac{3V}{4\pi}}$.

14. $s = \dfrac{Q}{W(t_1 - t_2)}$, $t_1 = \dfrac{Q}{Ws} + t_2$, $t_2 = t_1 - \dfrac{Q}{Ws}$.

15. $x = \dfrac{WsT + wt}{Ws + w}$, $w = \dfrac{Ws(T - x)}{x - t}$, $t = \dfrac{x(Ws + w) - WsT}{w}$, $s = \dfrac{w(x - t)}{W(T - x)}$.

16. $L = \dfrac{w(x - t)}{W} - (T - x)$, $x = \dfrac{W(L + T) + wt}{W + w}$.

17. $L = \dfrac{(w_1 + w_2s)(x - t)}{W} - (T - x)$, $s = \dfrac{W(L + T - x)}{w_2(x - t)} - \dfrac{w_1}{w_2}$,

$x = \dfrac{(w_1 + w_2s)t + W(L + T)}{w_1 + w_2s + W}$.

18. $a = \dfrac{L - l}{lt}$. **19.** $D = \dfrac{d}{1 + bt}$. **20.** $k = \dfrac{Hl}{A(T - t)}$, $T = \dfrac{Hl}{kA} + t$.

21. $a = \dfrac{2(s - ut)}{t^2}$, $u = \dfrac{s}{t} - \dfrac{1}{2}at$, $s = 16t^2$.

22. $s = \dfrac{v^2 - u^2}{2a}$, $a = \dfrac{v^2 - u^2}{2s}$, $v = \sqrt{u^2 + 2as}$. **23.** $v = \dfrac{Ft}{m} + u$.

24. $F = \dfrac{W}{2s}(v^2 - u^2)$, $g = \dfrac{v^2 - u^2}{2s}$. **26.** $m = \dfrac{FR^2r^2}{R^2 - r^2}$.

27. $f = \dfrac{uv}{u - v}$, $v = \dfrac{uf}{u + f}$. **28.** $+$ when $u < f$, $-$ when $f < u$.

29. $A = 1$, $B = 2$. **30.** $a = -124\frac{3}{8}$, $b = \frac{3}{16}$, $94 \cdot 4$.

31. $g = 4\pi^2 l/t^2$, $l = gt^2/4\pi^2 = 3 \cdot 26$.

Ex. XIV (c)

1. $\dfrac{88x}{3y}$. **2.** $\dfrac{88x - 3y}{60}$. **3.** $666 \cdot 4x$. **4.** $5 \cdot 25x$.
5. 9. **6.** $4\frac{5}{8}$, $7\frac{1}{8}$ in. **7.** 2, 24. **8.** 30 m.p.h.
9. 300 r.p.m. **10.** 9, 11, 13, 15. **11.** 3, 9. **12.** 21, 32.
13. 20×15 yd. **14.** 3. **15.** $57\frac{1}{2}$.

Ex. XV (A)

1. $x(x - 1)$. **2.** $a(1 - x)$. **3.** $a(a + x)$.
4. $ax(a - x)$. **5.** $x(x^3 + 1)$. **6.** $(2a + 3c)(a - 2b)$.
7. $(a - 3)(x - y)$. **8.** $(a^3 + 3)(a - 1)$. **9.** $(2y - 5)(x^2 + a)$.
10. $(x + 3)(x - a)$. **11.** $(a^3 - 2b^3)(a + b)$. **12.** $2x(x^2 - 4)(x - 3)$.
13. $x(a^4 + 4bc - b^2 - 4c^2)$.
14. (i) $(a - b)(a^2 + ab + b^2)$, (ii) $(a + b)(a^2 - ab + b^2)$.

Ex. XV (B)

1. $(x - 3)(x - 2)$. **2.** $(x + 3)(x + 2)$. **3.** $(x + 3)(x - 2)$.
4. $(x + 6)(x + 1)$. **5.** $(x - 3)(x + 2)$. **6.** $(x - 6)(x - 1)$.
7. $(x + 6)(x - 1)$. **8.** $(x - 6)(x + 1)$. **9.** $(a + 12)(a + 1)$.
10. $(a - 12)(a - 1)$. **11.** $(a + 12)(a - 1)$. **12.** $(a - 12)(a + 1)$.
13. $(a + 6)(a + 2)$. **14.** $(a - 6)(a - 2)$. **15.** $(a + 6)(a - 2)$.
16. $(a - 6)(a + 2)$. **17.** $(a + 4)(a + 3)$. **18.** $(a - 4)(a - 3)$.
19. $(a - 4)(a + 3)$. **20.** $(a + 4)(a - 3)$. **21.** $(x + y)(x - y)$.
22. $(x + 2y)(x - 2y)$. **23.** $(2x + y)(2x - y)$. **24.** $(x + 1)(x - 1)$.
25. $(xy + 1)(xy - 1)$. **26.** $(2a + 3b)(2a - 3b)$.
27. $(x + y - 1)(x - y + 1)$. **28.** $(x + y + 1)(x - y - 1)$.
29. $(3ax + 2by)(3ax - 2by)$. **30.** $(2a)(2x)$.
31. $(a - x + b + y)(a - x - b - y)$. **32.** $(a + x + b - y)(a + x - b + y)$.
33. $(x + y + 1)(x - y + 5)$. **34.** $(x + y + 1)(x - y - 5)$.

35. $(2x + 3y + 1)(2x - 3y + 5)$.
36. $(x^2 + y^2 + xy)(x^2 + y^2 - xy)$.
37. $(x^2 - y^2 + xy)(x^2 - y^2 - xy)$.
38. $(x - 4y)(x + 2y)$.
39. $(3x + 2y)(2x - 3y)$.
40. $3(2x - y)(x + 2y)$.
41. $(4x + 3y)(3x - 4y)$.
42. $(b - c)(b - c + 3)(b - c - 3)$.
43. $2(2x - y)(x + 2y)$.
44. $(a - b)(a + b + c)$.
45. $(a + b)(b + c)(c + a)$.
46. $3x(2a - x)(2a^2 - 2ax + 5x^2)$.

Ex. XV (c)

3. $(x - 1)(x - 2)(x + 3)(x + 4)$.
4. $(a + 1)(a - 2)(a - 3)(a - 4)$.
7. $ab(a - b) + bc(b - c) + ca(c - a)$, $a^2(b - c) + b^2(c - a) + c^2(a - b)$, $(a - b) + (b - c) + (c - d) + (d - a)$, $ab(b - c) + bc(c - d) + cd(d - a) + da(a - b)$, $\frac{a}{b - c} + \frac{b}{c - a} + \frac{c}{a - b}$.

Ex. XV (d)

1. 3, 2.
2. -3, -2.
3. $\frac{3}{2}$, 2.
4. 4, -3.
5. 6, -2.
6. 1, -15.
7. 8, 4.
8. -5, $\frac{1}{3}$.
9. 2, ± 3.
10. ± 2.
11. 3, -7.
12. -2.
13. 3, $\frac{2}{3}$, $-\frac{5}{2}$.
14. $2a$, a.
15. a, $-2a$.
16. $-\frac{2}{a}$, $-\frac{1}{a}$.
17. $(a - 1)$, $-(a + 1)$.
18. $\frac{3}{4}$, $\frac{2}{3}$.
19. a, b.
20. 2.

Ex. XV (e)

1. $(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.
2. $(x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$.
3. $(x - y)(x^2 + xy + y^2)$.
4. $(x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$.
5. $(x + y)(x - y)(x^2 + y^2)$.
6. $(x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6)$.
7. $(x - y)(x^2 + xy + y^2)(x^6 + x^3y^3 + y^6)$.
8. $x^3 - x^2y + xy^2 - y^3$.
9. $x^3 + x^2y + xy^2 + y^3$.
10. $8x^3 - 12x^2y + 18xy^2 - 27y^3$.
11. $4x^2 + 6xy + 9y^2$.
12. $x^6 + x^3y^3 + y^6$.
13. $x^4 + x^2y^2 + y^4$.
14. $x^3 + y^3$.
15. $a^4 + 2a^3b + 4a^2b^2 + 8ab^3 + 16b^4$.
16. $(x - 4)(x + 1)$.
17. $(R + r)(R^2 - Rr + r^2)$, $(R - r)(R^2 + Rr + r^2)$
18. $(a + b)^2 + 4(a + b)(c - d) + 16(c - d)^2$.
19. $(a - b)^3 - (a - b)^2(b - c) + (a - b)(b - c)^2 - (b - c)^3$.
20. $a^2 + 7b^2 + 4c^2 + 4ab - 2ac - 10bc$.

Ex. XV (f)

1. $(x - 4)$, $(x - 4)(x^2 - 1)$.
2. $(x + 3)$, $x(x + 3)(x - 4)(x - 1)$.
3. 2, $4x^2(x^2 - 2x - 3)$.
4. $(x^2 - 4)$, $(x + 1)(x^3 - x^2 - 4x + 4)$.
5. $(3a + 2)$, $(3a + 2)(2a - 1)(a + 1)(a^2 - a + 1)$.
6. $(a - b)$, $(a - b)^2(a + b)(a - 2b)$.
7. 1, $a^6 - b^6$.
9. $\frac{x + 1}{x - 1}$.

10. $\dfrac{2x - 3a}{4x^2 + 6ax + 9a^2}$. 11. $\dfrac{2x^2 - 1}{4x^2 + 3}$. 12. $\dfrac{(x - 4)(x - 7)}{x^2}$.

13. $\dfrac{(x - 1)(x - 2)}{x^3}$. 14. $\dfrac{x - a}{(a - b)(c - a)}$. 15. $-\dfrac{2}{x^2 - 4}$.

16. $\dfrac{a}{1 - a}$. 17. $\dfrac{x}{(x - y)^2}$. 18. $a^3 - 1$.

19. $\dfrac{4x^2 - 15x + 14}{1 - x^3}$. 20. $\dfrac{bx}{a - b}$. 21. $\dfrac{x + 3y}{x^3 - y^3}$.

22. $\dfrac{-12x}{x^3 + 2x^2 - 9x - 18}$. 23. (i) $\dfrac{(c - a)(a^2 + ab + b^2) - c^3}{(b - c)(c - a)}$, (ii) 0. 24. 1.

25. $\dfrac{(a^2 + b^2)^2}{a^4 + b^4}$. 26. $-\dfrac{y^2}{(x + y)^2}$. 28. 1. 29. −1.

30. $(a + b)$. 31. A = 3, B = 2.

Ex. XVI (A)

1·41421, 1·73205, 2·23607, 2·64575, 2·82843, 3·16228, 3·31662, 3·46410, 3·60555.

Ex. XVI (B)

2. $3\sqrt{3}$, $4\sqrt{3}$, $5\sqrt{5}$, $6\sqrt{7}$. 3. 2·828, 5·196, 6·928, 3·464, 6·708.
4. (i) $9\sqrt{2} - 7\sqrt{3}$, (ii) $5\sqrt{3} - 4\sqrt{5}$.
5. (i) 60, (ii) $\sqrt{30}$, (iii) $3 + \sqrt{6}$, (iv) 12. 6. $11\sqrt{6} - 2$.
7. $a + 2\sqrt{ab} + b$, $a - 2\sqrt{ab} + b$. 8. (i) $3(\sqrt{5} + \sqrt{2})$, (ii) $5\sqrt{3}$.
9. $11\sqrt{5}$. 10. (i) $2\sqrt{3}$, (ii) $\sqrt{3} - \sqrt{2} - 1$ or $1 + \sqrt{2} - \sqrt{3}$.
11. 10·806. 12. (i) 3·058, 0, (ii) ·1414, ·447, ·3146, 1·201.
13. $\dfrac{14 + 4\sqrt{6}}{11 - 6\sqrt{2}}$. 14. $(x + \sqrt{3})(x - \sqrt{3})$, $(x\sqrt{2} + \sqrt{3})(x\sqrt{2} - \sqrt{3})$.

Ex. XVI (C)

1. ·8944, ·577, 2·268, ·816. 2. 2·414, ·414, ·586. 3. ·892, 5·828.
4. $1\frac{1}{22}$. 5. 0. 6. (i) −6, (ii) $3a^2 - 2b^2$, (iii) $-3\frac{1}{4}$. 7. $8\sqrt{3}$.
8. (i) $3 + 2\sqrt{2} = 5·828$, (ii) $8 + 3\sqrt{6} = 15·35$, (iii) $\dfrac{24 - \sqrt{15}}{33} = ·61$.
(iv) $\dfrac{2\sqrt{5} + 3\sqrt{10} + 2\sqrt{6} + 4}{11} = 2·376$.
9. $-5\sqrt{2}$. 10. 5. 11. ·173. 12. $3\sqrt{2}$.

Ex. XVI (D)

1. $\sqrt{2} : 1 : 1$. 3. $5\sqrt{3}$, 5. 4. 800 sq. yd., $20\sqrt{2}$ yd. 6. $·289x$, $·0722x^2$.
7. $·155x$, $·0269x^2$. 8. $5\sqrt{2}$ in. 9. $\dfrac{5\sqrt{3}}{4}$ in.

Ex. XVII (A)

1. 1·4, 2·1, 1·2. **2.** 1·6, 3·6, −0·2. **3.** 1, 1·5.
4. 0·1, 2·1, 3·1. **5.** 0·5, 1·1, 0·2, 0·6, 1·9. **6.** 1·4, 2·4, 4·4.

Ex. XVII (B)

1. $a^{\frac{5}{3}}, a^{\frac{1}{2}}, a,^{\frac{1}{2}} a^{\frac{1}{3}}, (a+b)^{\frac{3}{2}}$. **2.** $a - a^{\frac{1}{3}}b^{\frac{1}{4}} + a^{\frac{2}{3}}b^{\frac{1}{2}} - b^{\frac{3}{4}}$. **3.** $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.
4. 0·4, 0·27, 0·1. **5.** 0·45, 0·2, 0·47.
6. 0·78, 0·18, 1·18, 0·9, 0·96, 1·44, 0·36, 0·7, 0·52, 1·52.

Ex. XVII (C)

1. $\overline{2}, \overline{4}, \overline{3}, \overline{1}, \overline{3}$. **2.** Nil, 3, 1, 5, 2, 7. **3.** 2·4915, 1·016, $\overline{1}$·2005, 0·299.
4. 1·48, $\overline{2}$·52, $\overline{4}$·234, 0·184. **5.** $\overline{16}$·21, $\overline{14}$·34. **6.** $\overline{1}$·3684, $\overline{2}$·3, $\overline{1}$·87.

Ex. XVII (D)

1. 0·2175, 1·2175, 2·2175, 3·2175, $\overline{1}$·2175, $\overline{2}$·2175, $\overline{3}$·2175.
2. 0·4771, 1·4771, 2·4771, $\overline{1}$·4771, 2·5159, 1·5159, $\overline{3}$·5159.
3. 1·9523, 3·9523, $\overline{1}$·9523, 3·7042, 0·6995, 0·4972.
4. 0, $\overline{1}$, $\overline{3}$, 1·0004, 2·0009, 3·0009.

Ex. XVII (E)

1. 1·473, 147·3, 0·01473, 0·1473. **2.** 36·92, 0·3692, 3692.
3. $\overline{1}$·2687, 0·1857. **4.** 1. **5.** $\overline{3}$·6358, 0·004323.
6. 0·05495. **7.** 0·1658, 0·02607. **8.** 0·1452.

Ex. XVII (F)

1. 5·933. **2.** 43·07. **3.** 106,900. **4.** 1·455. **5.** 177·4.
6. 3·729. **7.** 8·212. **8.** 27·49, 0·2749. **9.** 1·877. **10.** 2·975.
11. (i) 4·365, (ii) 0·6374, (iii) 1·315. **12.** 4·232. **13.** 4341.
14. 9·126. **15.** 0·419. **16.** 4·42. **17.** 4·767. **18.** 76·935.
19. 3·1827. **20.** $\overline{1}$·875.

Ex. XVIII

1. $2x^2 - 3x - 4$. **2.** $-x^2 + 6x - 3$. **3.** $2x^2 - 3x + 1$. **4.** $x^2 - 6$.
6. $z^2 + 2$. **8.** $x^2 + 2x + 3$. **9.** 3, $-\frac{1}{2}$. **11.** $W = 0{\cdot}785d^2$.
12. Ft. per sec. 48, 80, 113·1, 143; ft. 25, 100, $156\frac{1}{4}$, 289.
14. $z = x + 1$. **15.** 0. **18.** The values of y are equal at $x = \pm 2$.

Ex. XIX (A)

1. 3, -1. **2.** 2, -4. **3.** 0, -2. **4.** 3. **5.** 4, roots are equal.
6. -11, -10. **7.** $-\frac{3}{2}$, $-\frac{2}{3}$. **8.** 1, $-\frac{1}{6}$. **9.** 1, -3. **10.** $\frac{4}{3}$, $\frac{3}{4}$.
11. $\frac{7}{5}$, $-\frac{5}{2}$. **12.** 1, $-\frac{7}{4}$. **13.** 1, $-\frac{3}{7}$. **14.** $-\frac{1}{3}$, $-\frac{1}{4}$.
15. 1·66, $-$·361. **16.** $2 \pm 2\sqrt{3}$. **17.** $-4\frac{1}{2}$, -1. **18.** 4, $-\frac{5}{4}$.
19. $\dfrac{-5 \pm \sqrt{15}}{2}$. **20.** $\frac{1}{2}$, $-\frac{1}{4}$. **21.** $\dfrac{2 \pm 2\sqrt{2}}{3}$.
22. 1·34, -1·94. **23.** 5, -3. **24.** 1, $\frac{1}{2}$. **25.** $\frac{1}{2}$, $-\frac{4}{3}$.

Ex. XIX (B)

1. $4\sqrt{5}$ cm. **2.** $2\frac{1}{17}$ cm.
3. 96°, 148°; 100·8, 12·3 sq. cm.; 9·2, 4·44 sq. cm.
4. $3 \pm \sqrt{2}$ in. **5.** 2·77. **6.** 175 miles. **7.** 123 miles.
8. 2·84 miles.

Ex. XX (A)

14. Parabolic, (i) 2 miles, (ii) 12 miles. **15.** (i) 10 miles, (ii) 1 mile.
17. $\frac{5}{3}x^2 - 10x + 21$, $3 \pm 3\sqrt{-\frac{2}{5}}$. **18.** $3x^2 - 18x - 48$. **19.** For $> \dfrac{\sqrt{2}}{2}$.
21. $27x^2 - 34x + 3 = 0$.

Ex. XX (B)

1. ± 4, ± 2. **2.** ± 3, ± 2. **3.** $\pm\sqrt{\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$. **4.** 2, $\sqrt[3]{6}$.
5. 4, 3. **6.** $-2a$, a. **7.** Once, near -1·52; two.

Ex. XX (C)

2. $x = 4, -3\frac{4}{7}$. $y = 1, -2\frac{1}{35}$.
3. $x = 7, -4$. $y = 4, -7$.
4. $x = 0, 6$. $y = 0, 5$.
5. $x = 6, 3$. $y = \frac{3}{4}, \frac{1}{2}$.
6. $x = 4, 2, -2 \pm \sqrt{6}$. $y = 2, 4, -2 \mp \sqrt{6}$.
7. $x = \pm 9$. $y = \pm 4$.
12. $x = 2, -1·52$. $y = -1, 1·64$.
13. $x = \frac{1}{3}, \frac{1}{2}$. $y = -\frac{1}{2}, -1$.
14. $x = 1·175, -0·425$. $y = -0·65, -3·85$.
15. $x = \dfrac{3 \mp \sqrt{6}}{2}$. $y = \dfrac{\pm\sqrt{6} - 1}{2}$.
16. (i) $x = 5, 4$; $y = 4, 5$; (ii) $x = \pm 3$. $y = \mp 2$.
17. $x = 3·25, 1·92$. $y = 1·92, 3·25$.
18. (i) $\sqrt{7} + \sqrt{2}$, (ii) $\sqrt{7} - \sqrt{2}$. **19.** $3\sqrt{2} - \sqrt{5}$. **20.** $\sqrt{3} - \sqrt{2}$.
21. $\sqrt{19} + \sqrt{17}$. **22.** $\sqrt{a + b} + \sqrt{a - b}$. **23.** $\sqrt{\dfrac{a + b}{a - b}} - \sqrt{\dfrac{a}{b}}$.
24. 9 and 5. **25.** 14 and 6. **26.** 13 in.
27. $r = 5(2 + \sqrt{6})$. **28.** 1260 ft. (approx.). **29.** $12\frac{1}{2}$ and $22\frac{1}{2}$ m.p.h.
30. (i) -2 to $+2$, (ii) 2, -2, (iii) 2 to ∞ and -2 to $-\infty$.

Revision Ex. II

1. ·14%, ·4%. **2.** $y = 5 - 2x$. **3.** $y = \frac{\sqrt{3}x}{3} - 3$. **4.** 53, 22. **5.** 10.
6. (1) $(x+2)(x-1)$; (2) (i) $(x^2+2y^2)(x^2-8y^2)$, (ii) $(a+b)^3(a-b)$, (iii) $(x+1)(x-1)(2x-3y)$, (iv) $2(x-2)(x^3+2x^2+12)$.
7. -14. **8.** (i) $\sqrt{2}(4a-3)+\sqrt{3}(3b+2)$, $-5{\cdot}054$; (ii) $(x+y)(x^2+xy+y^2)$.
11. 3584. **12.** 1, 2, -3. **14.** $7+3x-2x^2$, $(\frac{3}{4}, 8\frac{1}{8})$.

Ex. XXI

2. 9·48 cm., 73°, 42°.
3. 41° 25′, 55° 46′, 82° 49′; 28° 57′, 46° 34′, 104° 29′.
4. 39·69 sq. cm., 46·48 sq. cm. **5.** $\frac{\sqrt{3}}{4}a^2$. **6.** 15·2, 7·6, 6·08 cm.

Ex. XXII (A)

1. $d = \frac{1}{2}at^2$. **2.** $d = ut - \frac{1}{2}at^2$. **3.** 16 ft., 576 ft., 1152 ft.
5. $h = \frac{1}{2}vt$, $t = \frac{v}{32}$. **6.** 18,000 ft.-lb., 4500 ft.-lb., 13,500 ft.-lb.
7. (i) $w = 5i$, (ii) $62\frac{1}{2}$ in.-lb., (iii) $w = 5(l-24)$.
8. $y = 425x + 14$ (approx.).

Ex. XXII (B)

1. 66π. **2.** 3·464 cm. **3.** $1013\frac{1}{3}\pi$ c.c. **4.** $\frac{7}{8}$.
9. 48π sq. in., 80π sq. in. **11.** $8\frac{1}{3}$ million sq. miles, 52 million sq. miles.
12. $E = 13{\cdot}42x - 0{\cdot}538x^2$ (approx.). **13.** $w = 0{\cdot}00026s^2 + 0{\cdot}17s$ (approx.).
14. $d = \frac{v^2}{64}$. **15.** 13·57 gall. **16.** $\theta = \frac{3}{2}C^2$. **17.** $d = 16t^2$.
18. $P = 40C - 2C^2$, $C = 10$. **19.** 7·938 cm. **20.** ·995 in.

Ex. XXII (C)

1. 1600 c. in., 1067 sq. in. **3.** 3944 sq. in., 7306 c. in.
4. 1018·3 sq. in., 763·7 c. in.

Ex. XXIII (A)

1. 2, $\frac{2}{\sqrt{3}}$, $\sqrt{3}$; $\sqrt{2}$, $\sqrt{2}$, 1; $\frac{2}{\sqrt{3}}$, 2, $\frac{1}{\sqrt{3}}$; 1, ∞, 0. **2.** Each = 1. **3.** 14·61.
5. $\frac{\sqrt{\sec^2 A - 1}}{\sec A}$. **6.** $\frac{3\sqrt{13}}{13}$, $\frac{5\sqrt{41}}{41}$. **9.** (i) $\frac{\sin A}{\sqrt{1-\sin^2 A}}$, (ii) $\frac{\sqrt{1-\cos^2 A}}{\cos A}$.

Ex. XXIII (B)

2. 50 lb., $50\sqrt{3}$. **3.** 163·84 lb. **4.** 511·4 ft. per sec.
5. 12·856 sq. in. **6.** $43{\cdot}58\pi$ sq. in.; 1743 in., 10 in.
7. (i) 105 m.p.h. N., (ii) 45 m.p.h. N., (iii) 93·68 m.p.h., 16° 6′ clockwise from N., (iv) 65·39 m.p.h., 53° 24′ clockwise from N.
8. 20·5 miles, 2·96 miles. **9.** 64·3° or 89° approx.
10. From 35° N. of E. (147·7 m.p.h.).
11. 1·4 miles horizontally. **12.** $y = \frac{2}{3}x - \frac{1}{18}x^2$.
13. $y = -0{\cdot}000213\dot{3}x^2 + 3{\cdot}2x$. **14.** 4500 ft. per sec.; if $y > x \tan e$.

Ex. XXIII (C)

4.

	120°	180°	210°	270°	300°	360°
sin	0·866	0	−0·5	−1	−0·866	0
cos	−0·5	−1	−0·866	0	0·5	1
tan	−1·7321	0	0·5774	∞	−1·7321	0
						etc.

5. 90°. **10.** Sine curve.

Ex. XXIII (D)

2. 0·3827, 0·9239, 0·4142. **3.** 0·9659, −0·2588, −3·7321.
4. (i) 45°; (ii) 0°, 90°, 180°; (iii) 0°; (iv) 45°, 135°.
5. $2x\sqrt{1-x^2}$, $1-2x^2$, $\sqrt{\dfrac{1-\sqrt{1-x^2}}{2}}$, $\sqrt{\dfrac{1+\sqrt{1-x^2}}{2}}$.
6. $4\sin^2 x - 4\sin^4 x$. **7.** $2\sin\dfrac{A}{2}\sqrt{1-\sin^2\dfrac{A}{2}}$.
14. (i) $2\sin A\cos B$, (ii) $2\cos A\sin B$, (iii) $2\cos A\cos B$, (iv) $-2\sin A\sin B$, (v) $\dfrac{2\tan A(1+\tan^2 B)}{1-\tan^2 A\tan^2 B}$, (vi) $\dfrac{2\tan B(1+\tan^2 A)}{1-\tan^2 A\tan^2 B}$.
15. 877 kilocycles. **16.** 90°, 53° 8′, 36° 52′.

Ex. XXIV (A)

1. (i) $\cos a = \cos b\cos c$, (ii) $\cos a = \cos b\cos c - \sin b\sin c = \cos(b+c)$
2. (i) 90°, (ii) 45°. **3.** A = 49° 33′, B = 71°, C = 74° 54′.
4. 2179 miles, 2209 miles, difference 30 land miles.
5. 4164 miles, 4418 miles, difference 254 land miles.
6. (i) 2911, (ii) 6790, (iii) 6120, (iv) 5860 sea miles.

Ex. XXIV (B)

2. (i) 115° 42′, (ii) 117° 18′, (iii) 126° 14′, (iv) 133° 6′ (approx.).
3. 113° 12′, 111° 6′, 97°, 79° 30′ (approx.).
6. 2782, 4300, 1400, 3260, 7460, 3360 miles (approx.).

Ex. XXV (A)

1. 3, 5, 23.
2. $2z^2 + 3z + 2$.
3. $f(y) = 4y^2 + 7y + 1$; $\frac{1}{4}$, -2.
4. $-2\frac{1}{3}$; $\frac{7}{3}$, -3.
5. $d = ut + \frac{1}{2}at^2$.
6. $\dfrac{2}{x+3} + \dfrac{1}{x+4}$.
7. $\dfrac{4}{2x+3} - \dfrac{3}{3x-2}$.
8. $\dfrac{3\sqrt{2}}{4}$.
9. 0·8037, 3.

Ex. XXV (B)

1. $a = \dfrac{b}{6}$, 0·75.
4. 7·56, 2·52 tons.
7. 5·06 in.
8. $n = 3{\cdot}534$, $k = 17{,}100$.
9. 478·5 million miles.
10. $30\frac{3}{8}$ sec., 11·312 in.

Ex. XXV (C)

1. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$,
$x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.
3. $a^3 - 3a^2b + 3ab^2 - b^3$, $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$, $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.
4. $a^{10} - 5a^8b^2 + 10a^6b^4 - 10a^4b^6 + 5a^2b^8 - b^{10}$.
5. $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$.
10. +32.
11. 1·04.

Ex. XXV (D)

1. 270·459 sq. in.
2. 113·427 sq. cm., 1703·822 c.c.
3. 250·675 c.c.
4. 3·03%.
5. 363·6 c.c.
6. 1·00006, 0·999, 10·007, 1·004, 0·984, 101·2.
7. 427° C. (approx.).
8. 974° C. (approx.).
10. 19·6 c. in.
11. (i) ·11, ·23, ·38; (ii) Four-figure tables show no difference.
12. (i) ·002945, (ii) ·001472, (iii) ·5; (i) 0, (ii) 0, (iii) ·5.

Ex. XXV (E)

1. $\frac{1}{2}\pi R^2$, πR^2.
2. $2\pi R^2$, $4\pi R^2$.
3. $\frac{4}{3}\pi a^3$.
4. $\frac{2}{3}\pi r^3$, $\frac{4}{3}\pi r^3$.
5. 0·8% (approx.).
6. 1·1%.
7. πRs.
8. $\frac{1}{3}\pi R^2h$, πR^2h.

Ex. XXVI (A)

1. (i) 20, 23, 26; (ii) -15, -20, -25; (iii) $19a$, $23a$, $27a$.
2. $3a$, $-a$, $-5a$, etc.
3. (i) 6, 10, 14, etc.; (ii) 6, 2, -2, etc.; (iii) -6, -2, 2, etc.; (iv) -6, -10, -14, etc.; (v) 1, -3, -7, etc.; (vi) 0, -2, -4, etc.
4. a, $(a + d)$, $(a + 2d)$, etc., one less.

Ex. XXVI (B)

1. (i) 486, 1458, 4374; (ii) $\frac{3}{64}$, $-\frac{3}{256}$, $\frac{3}{1024}$; (iii) $48a^5$, $96a^6$, $192a^7$.
2. $3a$, $-6a^2$, $12a^3$, etc.
3. (i) 1, -2, 4, -8, etc.; (ii) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.; (iii) -2, $\frac{2}{3}$, $-\frac{2}{9}$, etc.
4. a, ar, ar^2, etc., one less. **5.** (i) A.P., (ii) G.P., (iii) A.P., (iv) G.P.

Ex. XXVI (C)

1. -79. **2.** 2, 5, 8, 11, 14, 17. **4.** $6\frac{1}{4}$, $10\frac{1}{2}$, $14\frac{3}{4}$. **5.** 2, -1, -4, -7.
6. 325. **7.** $\frac{1}{2}n(n + 1)$. **8.** 400. **9.** 420.
10. n^2, $n(n + 1)$. **11.** 28. **12.** 1220.

Ex. XXVI (D)

1. 4374. **2.** -13122. **6.** $+12$, -48, $+192$ or -12, -48, -192.
7. 3 in. **8.** 728, -364. **9.** $11\frac{61}{64}$. **10.** $\frac{11605}{26244}$.
11. -3072. **13.** $\frac{173}{330}$, $\frac{461}{1998}$. **14.** $\frac{1}{1 + \delta}$.

Ex. XXVI (E)

1. (i) $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$; (ii) 4, ∞, -4. **3.** 4, 6.

Ex. XXVI (F)

1. £39. 8s. 0d. **3.** £265. 6s. 0d. **4.** 173,600. **5.** 2·022.
6. 2, 3·556, 6·234, 11·25, 20. **7.** 41·67. **8.** $111\frac{1}{9}$ yd.
9. £61. 15s. 0d.

Ex. XXVI (G)

1. (i) 1, (ii) -1, (iii) 0·699, (iv) $-0·301$.
2. (a) (i) 0·0218, (ii) 0·0214, (b) 3·9°.
3. G.P., ratio $\frac{2}{3}$. **4.** Equal. **5.** $1\frac{1}{2}$. **9.** $\frac{x}{n}$; after $n = x$.
10. £7. 7s. 0d., £137; £5. 18s. 0d., £172; £10. 16s. 0d., £95.
12. $\frac{a(1 - r^n)}{n(1 - r)}$. **13.** $a + \frac{n - 1}{2}d$. **14.** $\frac{3}{2} - \frac{1\frac{1}{2}}{3^x}$, $\frac{1\frac{1}{2}}{3^x}$.

15. 14·2 yr. **16.** $S_n = \frac{1 - x^n}{(1-x)^2} - \frac{nx^n}{1-x}$, 280483.

17. $\frac{x^n}{n}, x < \frac{n}{n-1}$.

Ex. XXVI (H)

1. 40320, 60, $\frac{1}{360}$, 5040, 10, 6, $\frac{4}{3}$, 2. **2.** ${}_{20}P_3$. **3.** ${}_{10}C_3$.
4. ${}_{52}P_5$, giving ${}_{52}C_5$ different hands. **5.** 360. **6.** 24.
7. (*a*) ${}_5P_3$, 100; (*b*) 85, 130. **8.** (*a*) 125, 180; (*b*) 155, 215. **9.** 11.

Ex. XXVI (I)

2. −448. **3.** 2^8. **4.** 3^5, 1. **5.** 10; 5th and 6th, 126. **6.** 8th, 15360.

Revision Ex. III

1. $1\frac{1}{4}$, $-2\frac{1}{4}$ (approx.). **3.** $\frac{\pi s^3}{24}(\sqrt{3} + 2)$.

6. 2·45 sec., 17·68 sec., 5·38 sec. **8.** $\frac{3\sqrt{2} - 4}{2}$, $(3\sqrt{2} - 4)$.

10. 1·264 in., 2·905 c. in.

Ex. XXVII (A)

1. 3. **2.** (i) 2, (ii) 2, (iii) −2, (iv) −2, (v) $\frac{1}{4}$, (vi) $\frac{1}{4}$. **3.** 0.

Ex. XXVII (B)

1. −12, −8, −2, 0, 2, 8, 12. **2.** 0.

Ex. XXVII (C)

1. (i) 1, ·866, ·707, ·5, 0, −·5; (ii) 0, −·5, −·707, −·866, −1, −·866; (iii) 1, $1\frac{1}{3}$, 2, 4, ∞, 4.
3. 2500, 2496.
4. (i) ·293, (ii) ·707, (iii) ·366, (iv) ·366, (v) ·423.
6. (i) ·293, (ii) ·366, (iii) 1.
7. (i) $\mp\frac{3}{4}$, (ii) $\mp\frac{4}{3}$, (iii) ∞, (iv) 0. **8.** 1.

Ex. XXVII (D)

1. $\frac{x\cos x - \sin x}{x^2}$. **2.** $\frac{\sin x - x\cos x}{\sin^2 x}$. **3.** $-\operatorname{cosec}^2 x$.

4. $\frac{2x\cos x + x^2\sin x}{\cos^2 x}$. **5.** $-\frac{x\sin x + 2\cos x}{x^3}$. **6.** $-\frac{nk}{x^{n+1}}$.

7. knx^{n-1}. **8.** $\sin 2x$. **9.** $-\sin 2x$.

10. $-\frac{1}{3}x^{-\frac{4}{3}}$ **11.** $2x\sin x + x^2\cos x$.

12. $\sin x \cos x + x(\cos^2 x - \sin^2 x) = \frac{1}{2}\sin 2x + x \cos 2x$.
13. $n \cos x \sin^{(n-1)} x$. **14.** $-n \sin x \cos^{(n-1)} x$.

Ex. XXVII (E)

1. $6(2x - 3)^2$. **2.** $8(3x + 1)(3x^2 + 2x - 3)^3$. **3.** $2x/\sqrt{2x^2 - 1}$.
4. $-x/\sqrt{a^2 - x^2}$. **5.** $x/\sqrt{(a^2 - x^2)^3}$.
6. $3 \sin^2 x \cos x$, $\frac{1}{2} \cos \frac{1}{2}x$, $-\frac{1}{2} \sin \frac{1}{2}x$.
7. (i) $x/\sqrt{a^2 + x^2}$, (ii) $2\sqrt[3]{a/9x}$, (iii) $\dfrac{bx}{a}\Big/\sqrt{x^2 - a^2}$, (iv) $-x/\sqrt{a^2 - x^2}$,

(v) $\dfrac{1}{9\sqrt[3]{x^2}} + \dfrac{1}{\sqrt{x^3}}$, (vi) $\frac{1}{2}\sec^2 \frac{1}{2}x$.

Ex. XXVII (F)

3. 1·7918. **6.** $\sqrt[3]{\dfrac{3}{x^2(2x + 3)^4}}$.

8. (i) $\dfrac{1}{k + x}$, (ii) $-\dfrac{1}{k - x}$, (iii) $-3e^{-3x}$, (iv) $3^x \log 3$.

9. $-\frac{1}{3}e^{-3x}$, $\dfrac{3^x}{\log_e 3}$, $\dfrac{\pi}{2}$.

10. 2^n. **11.** $k = \cdot 011$, 32·4 millions. **12.** ·0488.
13. £2·718, £2. **14.** 614·4 gall., 164 min. ($k = \cdot 04462$).

15. 0·0603. **16.** (i) $\cos x\, e^{\sin x}$, (ii) $\tan^{-1} x + \dfrac{x}{1 + x^2}$.

Ex. XXVIII (A)

1. 0·975, −4·09, −2·88. **2.** $2\frac{1}{17}$. **3.** $-1\frac{4}{7}$, $-3\frac{1}{11}$.
4. 22° 48′. **5.** 0·7030. **6.** 33° 41′.
7. $1 + 4\sqrt{2}$, $x = \sqrt{2}$. **8.** 15° or 89° (approx.). **10.** $\frac{1}{4}p \times \frac{1}{4}p$.

11. $l = 3$ ft., $r = \dfrac{3}{\pi}$ ft.

Ex. XXVIII (B)

1. $15\frac{5}{7}$ c.c., $6\frac{2}{7}$ sq. cm. **2.** 110 sq. cm. **3.** 3,127,000 sq. miles.
4. 0·982 c. ft. **5.** 1·84 c. ft.

Ex. XXVIII (C)

1. 0·8 c. ft. = 5 gall.

2. $t = \dfrac{\pi k^2 x^2}{2c} + C$, where k = radius : depth of vessel.

3. (i) $t = \dfrac{2\pi k^2 x^{2\cdot 5}}{5c} + C$, (ii) $t = \dfrac{\pi k^2}{c}x + C$. **4.** 121·7 sec.

Ex. XXIX (A)

2. $\frac{\sqrt{21}}{5}, \frac{\sqrt{29}}{5}$. **3.** $x = \pm 4, y = 0$. **4.** $x = 1\frac{1}{3}, y = \pm 2\sqrt{2}$.

5. $e = \sqrt{5}/3 = \cdot 745$, foci ·56 in., directrices 1·006 in. from centre.

Ex. XXIX (B)

1. $-\frac{a^2}{b^2}\frac{y_1^2}{x_1}, -\frac{b^2}{a^2}x_1, +\frac{a^2}{b^2}\frac{y_1^2}{x_1}, +\frac{b^2}{a^2}x_1$.

3. $x = 4, y = \pm 6\sqrt{3}$. **5.** $48\frac{1}{2}°, 131\frac{1}{2}°$ (approx.).

Revision Ex. IV

1. 2852 sea miles. **2.** (i) ·58. **3.** ·2865. **4.** 23·14 lb. **5.** ·475.

7. $\frac{-x}{\sqrt{(a^2 - x^2)}}$. **9.** $\frac{1}{2}x - \frac{1}{4}\sin 2x$. **12.** $x = 4, y = \pm 3\sqrt{3}$; 30π.

13. $y = \sqrt{3}(x - 1), y = \frac{3\sqrt{3}}{8}(x + 4)$; 27°. **14.** $\frac{\pi}{2}(2a^2 + b^2)$.

15. 11π c. ft.